MATHEMATICS MAGAZINE

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MORE PROGRESS TO MADNESS VIA "EIGHT BLOCKS"

ANDREW SOBCZYK, Clemson University

1. Introduction. Reference is made to the paper by S. J. Kahan, "*Eight Blocks to Madness*"—*A Logical Solution*, published in the issue for March 1972 of this MAGAZINE. Eight cubical blocks, each having sides of six different colors, are to be fitted together to form an outer cube, such that for each face of the outer cube, the four square faces which form the face are of the same color, and such that the six faces also are of the six different colors. Any particular choice of eight (repetitions allowed) of the thirty possible types of blocks (see section 2) will be referred to as a *puzzle*. We offer here general results relating to the class of puzzles, including particularly a Chart of Incompatibilities, and a List of Ground Layers, by means of which short work is made of finding the solution or solutions of any given puzzle. To illustrate the general procedure, and to illuminate the large variety of kinds of puzzles, in section 5 several examples are treated, including the specific puzzle which was discussed by Kahan, and a puzzle which has six different outer cubes (o.c.'s) as possible solutions. It is shown that no puzzle can have more than six o.c. solutions.

2. Thirty possible types of blocks or dice. A type of block is described by specifiying the colors of its Ceiling (C), Ground (G), Left face (L), Right face (R), Front (F), and Back (B). By rotation, each possible type of block may be placed so that it has a given fixed one of the six colors for C. Then there are five possible colors for G. Then for each color of G, there are six possible trace squares of a horizontal plane with the block, with edges of the four remaining colors. Therefore it is clear that there are thirty different types of blocks (and of o.c.'s).

With Kahan, we choose as six definite colors for the faces of the block, white, yellow, blue, green, purple, red ($w, y, b, g, p, r,$), and to facilitate designation of the various types, we let each cube be rotated so that it has its white face in the ceiling position. In Kahan's list (*loc. cit.*, p. 59), types 1 through 6 have ground yellow, 7 through 12 purple, 13 through 18 red, 19 through 24 green, and 25 through 30 blue. If we indicate the color of ground by a corresponding capital letter, and L, R and F, B colors by respective juxtaposed small letters, then Kahan's thirty types in numerical order are $Y(pg, rb)$, (pg, br), (pb, gr), (pb, rg), (pr, bg), (pr, gb), $P(yg, br)$ (yg, rb), (yb, rg), (yb, gr), (yr, bg), (yr, gb), $R(gb, py)$, (gb, yp), (gy, bp), $(gy; pb)$, (gp, by), (gp, yb), $G(rb, py)$, (rb, yp), (ry, bp), (ry, pb), (rp, by), (rp, yb), $B(ry, gp)$, (ry, pg), (rp, gy), (rp, yg), (rg, yp), (rg, py). Successive odd, even numbered types are *antitypes*, i.e., either is obtained from the other by interchange of the colors of one pair of parallel faces. The effect of interchange of colors in two pairs of parallel faces is to return to the original type. Interchange of colors in three pairs of course yields the antitype.

For reasons of symmetry, we designate the thirty types, consisting of fifteen antitypical pairs of blocks, as follows. If, e.g., Y_i is a type, then Y_i' is its antitype, and the antitype of Y_i' is Y_i. Our types are Y_1 = Kahan no. 1 = $Y(pg, rb)$, Y_2 = Kahan no. 4 = $Y(pb,rg)$, Y_3 = no. 6 = $Y(pr,gb)$, P_1 = no. 9 = $P(yb,rg)$, P_2 = no. 7 = $P(yg,br)$, P_3 = no. 12 = $P(yr,gb)$, R_1 = no. 13 = $R(gb,py)$, R_2 = no. 16 = $R(gy,pb)$, R_3 = no. 17 = $R(gp,by)$, G_1 = no. 22 = $G(ry,pb)$, G_2 = no. 20 = $G(rb,yp)$, G_3 = no. 23 = $G(rp,by)$, B_1 = no. 28 = $B(rp,yg)$, B_2 = no. 29 = $B(rg,yp)$, B_3 = no. 26 = $B(ry,pg)$, and the antitypes of each of the fifteen foregoing, designated by the corresponding primed symbols. For quick identification of each of the thirty types, reference may be made to the outer squares in Figure 2, below. The outer squares are the trace squares for Y_1 through B_3, in the same order as that in which they have just been listed.

Since each type is a mirror-image of its antitype, it is clear that no type of block will fit, at any corner, into its antitype as an o.c. Thus the antitypes of the blocks in any puzzle all are excluded as possible o.c.'s for a solution. Say that two blocks in an assembled o.c. are in *facially adjacent* position if they have a face in common. In any solution of a puzzle which contains an antitypical pair of blocks, the pair cannot be in facially adjacent position. For if they were, the o.c. would have to have two sides of the same color, which is not allowed.

Generally if two types are such that one as a block will not fit, at any corner, into the other as o.c., then the types will be called *incompatible.* Clearly incompatibility is a mutual relationship. The presence of a type of block in a puzzle excludes all incompatible types as possible o.c.'s for a solution. A necessary condition for the existence of a solution is that there be at least one type of o.c. with which all eight blocks are compatible.

3. Puzzles without solution, and with one or two solutions. It is obvious that a puzzle which consists of eight copies of one identical type of block is trivial: there is one and only one type of outer cube for a solution, namely the common type of the eight blocks.

THEOREM 1. *Any puzzle which contains three or more copies of one type uniquely must have that type of o.c. for any possible solution.*

Proof. Refer to the two possible diagonal locations of two blocks as *facially diagonal* (e.g., if both are in diagonal position within the ceiling layer of four blocks), and *cubically* (or *spatially*) *diagonal.* Obviously if two blocks are in either facially adjacent or cubically diagonal position, the placement of a third block in the o.c. will cause two of the three to be in facially diagonal position. The requirement that the faces of the o.c. be of the six *different* colors forces the type of the o.c. for any possible solution to be the common type.

COROLLARY 2. *Any puzzle which contains three copies of each of two different types of block has no possible solution.*

For of course the o.c. for a solution cannot simultaneously be of two different types. Corollary 2 implies that any puzzle which contains only three different types of blocks has no solution unless at least four of the blocks are of one of the types. Distributions 1, 3, 4 and 2, 3, 3 have no solution; distribution 2,2,4 can have at most one o.c. for a solution, the one of the type of the four like blocks.

COROLLARY 3. *If a puzzle contains three like blocks, and any block of type incompatible with the common type of the three (e.g., the antitype), then the puzzle has no solution.*

As shown by Kahan, the specific puzzle $(G_2, R_3, B_3', P_2', R_2', P_3', G_3, B_2)$ has only one solution for o.c., (with two different possible placements of the blocks in the o.c.). By consideration of trace squares, it may be seen that any puzzle which consists of four pairs of different Y_i's and Y_i''s (e.g., Y_1, Y_1, Y_2, Y_2, Y_3, Y_3, Y_1', Y_1') has at least two different solutions for o.c., and it may be shown that such a puzzle has exactly two solutions. (The author also has found puzzles which have exactly three, four, five, and six different o.c. solutions. Some of these will be exhibited in section 5.)

4. The one-parallel and no-parallel relations. For the thirty types of blocks, we define an irreflexive, symmetric (partial) relation as follows. Two different, non-antitypical blocks are *one-parallel* if they have one pair of parallel faces of the same colors, e.g., if they can be placed so that both have white C, yellow G; or so that both have red L, blue R. Otherwise the pair of blocks is in the *no-parallel* relation.

THEOREM 4. *If a block is in the one-parallel relation to an o.c., then it fits that o.c., but only in either of two facially adjacent positions.*

Proof. Suppose, e.g., that the block and the o.c. have the same colors for C and G. Two different nonantitypical trace squares fit each other at one and only one corner. Therefore the block fits the o.c. only along that corner, in either the C or G layer.

THEOREM 5. *If a block is in the no-parallel relation to an o.c., then either it is incompatible with that o.c. (i.e., fits at no corner of the o.c.), or it fits into the o.c. only in either of two cubically diagonal positions. In the former case, the antitypical block will fit the o.c. in either of two cubically diagonal positions. One and only one of an antitypical pair of blocks is compatible with a no-parallel related outer cube.*

Proof. Comparison of Y_1 with any no-parallel related block, e.g., P_3, shows either that at a corner where the three colors are common, in one they are clockwise (c.) and in the other counterclockwise (c.c.) (e.g., the respective c. and c.c. *wgr* corners of Y_1 and P_3), or that they agree at a corner (e.g., at c. *wgr* for Y_1 and P_3'). At the diagonally opposite corners where the colors also agree, the situation is the same (for Y_1 and P_3, the *ybp* corner is respectively c. and c.c., while for Y_1 and P_3', both are c.). Change from c. to c.c. at only one of the two diagonally opposite corners forces the two blocks to be in the one-parallel relation.

Table 1 below presents the one-parallel relation, and also the relation of incompatibility, for the thirty types of blocks. The one-parallel relation is indicated by $\cdot$, and incompatibility by $\times$. We ask the reader to bear in mind that, e.g., $Y_1 \cdot G_2$ implies $Y_1' \cdot G_2$, $Y_1 \cdot G_2'$, $Y_1' \cdot G_2'$, and also that, e.g., $Y_1 \times P_3$ is equivalent to $Y_1' \times P_3'$, $Y_1' \times B_2$ to $Y_1 \times B_2'$. Use of these facts, and of symmetry of the relations, reduces the size of Table 1 by a factor of one-fourth.

TABLE 1

The one-parallel relation, and incompatibility

	Y_1	B_1	G_1	P_1	R_1	R_2	Y_2	B_2	G_2	P_2	P_3	R_3	Y_3	B_3	G_3	
Y_1		×	×	×	×	×	·		·	·	×	·	·	·		Y_1
B_1			×	×	×	·	×	·		·		×	·	·	·	B_1
G_1				×	×	·	·	×	·		·		×	·	·	G_1
P_1					×		·	·	×	·	·	·		×	·	P_1
R_1						·		·	·	×	·	·	·		×	R_1
R_2		·	·	×	·		·			·	×	·	×			R_2
Y_2	·		·	·	×	·		·				×	·	×		Y_2
B_2	×	·		·	·	×	·		·				×	·	×	B_2
G_2	·	×	·		·	×	×	·		·	×			×	·	G_2
P_2	·	·	×	·		·	×	×	·		·	×			×	P_2
P_3		×	·	·	·		×	×		·			·	·		P_3
R_3	·		×	·	·	·		×	×		×			·	·	R_3
Y_3	·	·		×	·		·		×	×	·	×			·	Y_3
B_3	·	·	·		×	×		·		×	·	·	×			B_3
G_3	×	·	·	·		×	×		·		×	·	·	×		G_3
	Y_1'	B_1'	G_1'	P_1'	R_1'	R_2'	Y_2'	B_2'	G_2'	P_2'	P_3'	R_3'	Y_3'	B_3'	G_3'	

As a result of (hard-won) symmetrical design by the author, the five blocks Y_1, B_1, G_1, P_1, R_1 are *mutually* no-parallel and *mutually* incompatible. Interchange of colors b, g yields the same situation for the five blocks Y_2, G_3', B_3, P_2', R_1'; of r, g, the same for Y_3, B_3', R_2, P_1', G_2'; of g, p, the same for Y_1', B_2, P_3', G_3, R_2'; of g, y the same for G_2, B_1', Y_2', P_3, R_3'; and of r, y, the same for R_3, B_2', G_1', P_2, Y_3'. This partitions the set of thirty (types of) blocks into six pairwise disjoint subsets, each containing five blocks. Passage to the antitypes of the blocks of each set of five yields another partitioning of the thirty blocks into six sets, each consisting of five mutually incompatible blocks. These incompatibility relations include all of the (nonantitypical) incompatibilities between the thirty blocks. All of this information is presented in the simple chart on page 120 (Figure 1).

The presence of a given block in a puzzle excludes the four other blocks in the column containing the given block, and excludes the four other blocks in the polygonal line containing the given block, as possible o.c.'s for a solution of the puzzle. As already remarked, the given block also excludes its antitype. Thus the Chart (Figure 1) is a simplified representation of the graph of the relation of incompatibility, and in graph-theoretical language, clearly that graph is a regular graph on thirty vertices, of valency nine (i.e., each type is incompatible with its antitype, and with eight other types).

5. Method for solution of any puzzle. The following list of fifteen squares, with marked subsquares, presents information gained by use of Table 1, in conjunction with Theorems 4 and 5, and with the description of the thirty types in section 2. With the types in the standard positions as in section 2, the outer squares in the five lines in left to right; top to bottom order represent the *Ground* layers of, respectively, Y_1, Y_2, Y_3; P_1, P_2, P_3; R_1, R_2, R_3; G_1, G_2, G_3; B_1. B_2, B_3. For each o.c., the blocks which will fit into the ground layer and where, are indicated by the marked types on the subsquares. The squares represent the *Ceiling* layers for o.c.'s which are of the antitypes Y_1' Y_2', etc,, provided it is understood (i) that the antitypical o.c.'s are in inverted position so that ceiling and ground colors are interchanged (thus preserving the indicated colors of the vertical sides), and (ii) that each type marked on a subsquare is replaced by its antitype. The lower two blocks in a corner represented by a subsquare also fit in the *cubically diagonally located* corner of the other layer (ground or ceiling). The antitypes of the upper three blocks in the subsquare are those which fit in the cubically diagonally located corner. The upper three blocks are one-parallel with the type which is represented by the outer square; the other two blocks are no-parallel related with that type.

Now we are ready to present our method or procedure for dealing with any puzzle. As a first example, we solve the specific puzzle which was discussed by Kahan.

STEP 0. *By reference to the outer squares of Figure 2, determine which types of blocks are included in the given puzzle, and how many of each.*

By examination and comparison of the blocks of the specific puzzle with Figure 2, it is determined that they are P_2', P_3', R_2', R_3, G_2, G_3^-, B_2, B_3'.

STEP 1. *In case the puzzle contains three or more copies of one type X, then by Theorem* 1, *X is the only possible o.c. for a solution. If the puzzle also contains one X', or three copies of another type Y (as it would, e.g., if only two types were represented in the puzzle), then by Corollaries* 2–3, *there is no solution. If there is neither an X' nor three Y's, then refer to the Chart (Step* 2) *to see if X is excluded as o.c. by another block in the puzzle. If not, proceed to Step* 3. (As a consequence of the remark about distributions following Corollary 2, in order that a puzzle possibly may have more than one o.c. solution, it is necessary that it contain four or more different types.)

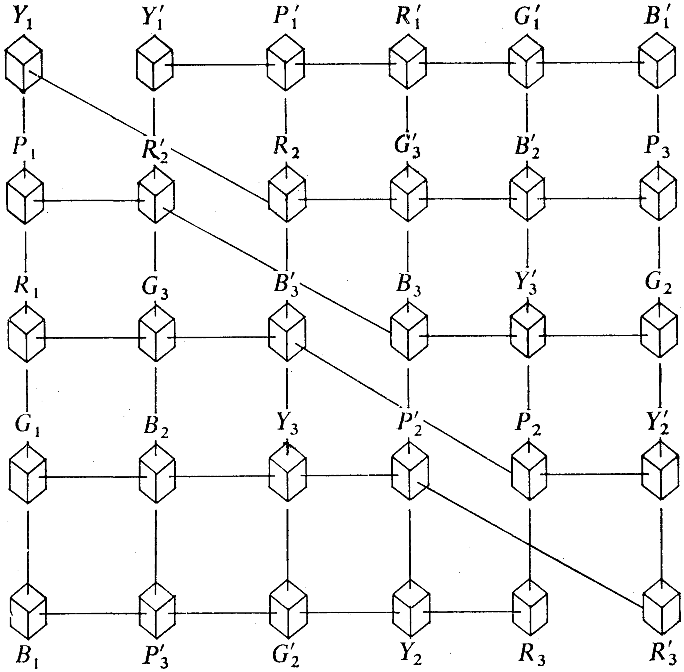


FIG. 1.

Chart of Incompatibilities

The blocks in each column of five blocks are *mutually* incompatible. The set of five joined blocks in the top row, the set of five in the bottom row, and the sets of five in the four bent rows, likewise each are mutually incompatible sets. All nonantitypical incompatibilities, one-hundred twenty in number, are included among those of the mutual incompatibilities (ten per each of the twelve sets of five). Thus the Chart is a simple representation of a regular graph of valency eight (nine if the antitypicalities are included) on thirty vertices (one-hundred thirty-five edges).

Notes for FIG. 2, **List of Ground Layers** (page 121)

The possible ground or ceiling layers, for any typical or antitypical o.c., may be derived from the squares in Figure 4. As marked, the squares represent the ground layers for Y_1, Y_2 Y_3; P_1, P_2, P_3; etc. With the marked types replaced by their antitypes (e.g., Y_3 by Y'_3, Y'_2 by Y_2, etc.), the same squares represent the ceiling layers for o.c.'s Y'_1, Y'_2, Y'_3, etc., in inverted position. The square for the ceiling layer of e.g.o.c. Y_1 is obtained by marking Y'_3 P'_2 B'_3 in, e.g., the subsquare diagonally opposite the subsquare which is marked $Y_3 P_2 B_3$, and P'_3 B'_1 in the same diagonally opposite subsquare. The same new square, except with all marked types replaced by antitypes in the same positions, represents the ground layer of inverted o.c. Y'_1. Each type of block fits at every corner of the same type of o.c.; the types identical with the o.c. types are not marked in the subsquares.

	b		
p	$Y_3P_2B_3$ $P'_3B'_1$	$Y'_2G'_2B_3$ G'_1B_2	g
	$Y_2P_2R'_3$ $P'_1R'_2$	$Y'_3R'_3G'_2$ R'_1G_3	
	r		

	g		
p	$Y'_3P'_1G_1$ P_3G_3	$Y'_1G_1B'_2$ $G_2B'_3$	b
	$Y_1P'_1R'_2$ $P_2R'_3$	$Y_3R'_2B'_2$ $R_1B'_1$	
	r		

	b		
p	$Y_1P'_3B'_1$ P_2B_3	$Y_2R_1B'_1$ $R'_2B'_2$	r
	$Y'_2P'_3G'_3$ $P_1G'_1$	$Y'_1R_1G'_3$ R_3G_2	
	g		

	g		
y	$Y'_2P'_3G'_3$ $Y_3G'_1$	$P_2G'_3B_2$ $G'_2B'_1$	b
	$Y'_2P'_2R_3$ Y'_1R_2	$P_3R_3B_2$ $R'_1B'_3$	
	r		

	r		
y	$Y_1P'_1R'_2$ $Y_2R'_3$	$P_3R'_2G'_2$ R'_1G_1	g
	$Y_1P'_3B'_1$ Y_3B_3	$P_1G'_2B'_1$ G'_3B_2	
	b		

	b		
y	$Y'_3P'_2B'_3$ Y'_1B_1	$P_1R'_1B'_3$ R_3B_2	r
	$Y'_3P'_1G_1$ Y_2G_3	$P_2R'_1G_1$ $R'_2G'_2$	
	g		

	y		
g	$Y_3R_3G_2$ $Y'_1G'_3$	$Y_3R'_2B'_2$ $Y_2B'_1$	b
	$P'_3R_2G_2$ $P'_2G'_1$	$P'_3R'_3B'_2$ P'_1B_3	
	p		

	b		
g	$R'_3G'_1B_1$ G_3B_3	$Y'_2R'_1B_1$ Y'_3B_2	y
	$P'_2R_1G'_1$ P'_3G_2	$Y'_2P'_2R_3$ Y'_1P_1	
	p		

	y		
g	$Y'_1R_1G'_3$ Y_3G_2	$Y'_1P_1R_2$ $Y'_2P'_2$	p
	$R'_2G'_3B'_3$ $G_1B'_1$	$P_1R'_1B'_3$ P_3B_2	
	b		

	b		
r	$R'_2G'_3B'_3$ $R_3B'_1$	$Y_2G_2B'_3$ $Y'_1B'_2$	y
	$P_3R'_2G'_2$ $P_2R'_1$	$Y_2P_3G_3$ $Y'_3P'_1$	
	p		

	p		
r	$P'_2R_1G'_1$ P'_3R_2	$P'_2G_3B'_2$ P'_1B_1	b
	$Y'_1R_1G'_2$ Y_3R_3	$Y'_1G_1B'_2$ $Y_2B'_3$	
	y		

	y		
r	$Y'_3R'_3G'_2$ $Y_1R'_1$	$Y'_3P'_1G_1$ Y_2P_3	p
	$R'_3G'_1B_1$ R_2B_3	$P'_1G_2B_1$ $P'_2B'_2$	
	b		

	g		
r	$R_2G_3B_3$ $R'_3G'_1$	$P'_2G_3B'_2$ P'_1G_2	p
	$Y'_3R_2B_2$ $Y'_2R'_1$	$Y'_3P'_2B'_3$ Y'_1P_3	
	y		

	p		
r	$P_1R'_1B'_3$ P_3R_3	$P_1G'_2B'_1$ $P_2G'_3$	g
	$Y'_2R'_1B_1$ Y'_3R_2	$Y'_2G'_2B_3$ $Y_1G'_1$	
	y		

	g		
r	$R'_3G'_1B_1$ R_2G_3	$Y_1G'_1B_2$ $Y'_2G'_2$	y
	$P'_3R'_3B'_2$ P'_1R_1	$Y_1P'_3B'_1$ Y_3P_2	
	p		

In the specific puzzle, there is no repetition of types. Therefore in order to see which types are possible o.c. solutions for it, proceed to Step 2.

STEP 2. *Mark five rows of six dots, as in the chart (Figure 1) on a piece of paper, and circle those corresponding to the types which occur in the puzzle. Draw in the vertical and the polygonal lines of the chart which pass through the circled dots. If two or more circled dots are in a line then all types in that line are excluded. In case of only the one circled dot in both of the lines through it, the corresponding type is a possible o.c. unless it is excluded by an antitype. Mark × through any antitypes which are not already excluded by the drawn-in lines.*

Following (Figure 3) is the result of the procedure of Step 2 applied to the specific puzzle. All eight antitypes already are excluded by the other incompatibilities, so in this case of the specific puzzle, it is not necessary to mark any antitypes.

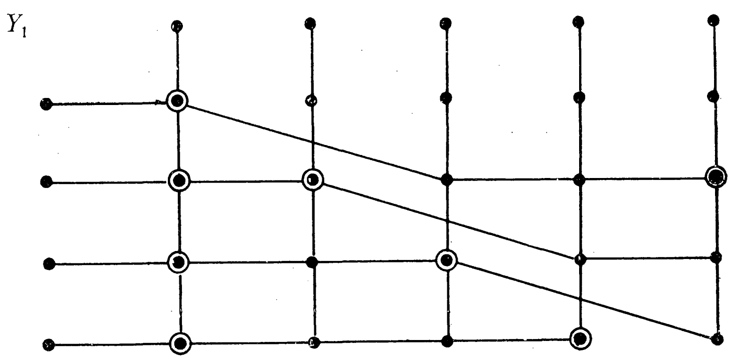


FIG. 3. **Chart for the specific puzzle.**

This shows that all o.c.'s except Y_1 are excluded so that o.c. Y_1 is the only possible solution.

It is obvious that Y_1 is the only possible o.c. solution for the specific puzzle. (It is also obvious that the specific puzzle contains four mutually incompatible types, and that any puzzle which contains five mutually incompatible types has no solution. However if, e.g., the five types are those in the second column then the exclusion of Y_1, R_2, G_3', B_2', P_3 is only by the fact that they are the antitypes of the types in the second column.)

STEP 3. *For each possible type of o.c. which is yielded by the chart, by using the List of Ground Layers, sketch the outer squares for the ground and ceiling layers, marking down in the appropriate subsquares those types which are included in the puzzle. If a simultaneous placement of the eight blocks in the eight subsquares* *is possible, then that o.c. is a solution, with the indicated placement.* Otherwise of course the o.c. is not a solution in spite of the individual compatibility of the blocks with the o.c. For a given o.c. solution there may exist several alternative placements.

Applied to the specific puzzle, Step 3 produces the three squares in Figure 4. Placement in the ground layer is unique; the other two squares represent the possible alternative placements in the ceiling layer.

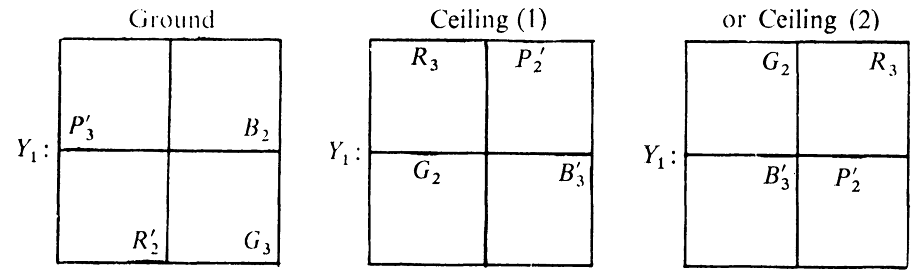


FIG. 4. **Placements of the blocks for the specific puzzle (Example 1).**

EXAMPLE 2. Consider puzzle $\mathscr{P}_2 = (Y_2, Y'_3, P'_1, P_2, R'_2, G'_2, B'_1, R'_3)$. As in the case of the specific puzzle $\mathscr{P}_1$, use of the chart shows that Y_1 is the only possible o.c. for a solution. See Figure 5. But reference to the list reveals that no block of $\mathscr{P}_2$ fits in subsquare (or corner) *CLB* of o.c. Y_1. Therefore $\mathscr{P}_2$ has no solution. There are, however, several simultaneous and compatible placements of seven of the blocks in o.c. Y_1.

EXAMPLE 3. Consider puzzle $\mathscr{P}_3 = (P_2, P_2, R'_2, G'_1, G_2, G_3, G_3, B'_1)$. This puzzle contains six different types, but does not contain three of a common type. Therefore we proceed to Step 2. The marked chart is as follows. See Figure 6. All o.c.'s evidently are excluded, except Y_1, Y_2, Y_3. Proceeding to Step 3, we find that in fact each of Y_1, Y_2, Y_3 is a solution of $\mathscr{P}_3$. The ground and ceiling layers are as follows. See Figure 7. In the case of each of the three o.c. solutions Y_1, Y_2, Y_3, the placement of the blocks is uniquely determined.

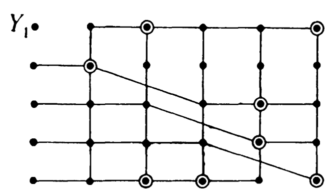


FIG. 5
Chart for Puzzle $\mathscr{P}_2$

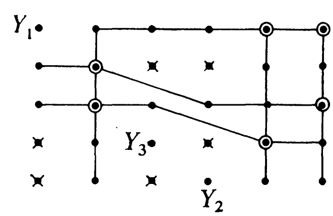


FIG. 6
Chart for Puzzle $\mathscr{P}_3$

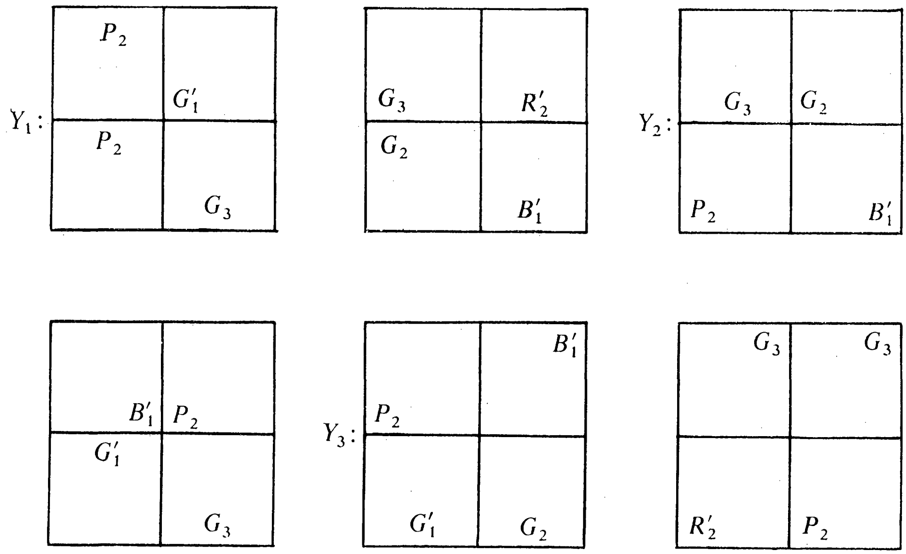


FIG. 7
Unique placements for the three o. c. solutions of $\mathscr{P}_3$.

For further examples, the reader easily may determine, using the chart and the list, that puzzles $\mathscr{P}_4 = (Y_3, P_3', R_1, G_3, B_1', B_1', B_2', B_3)$, $\mathscr{P}_5 = (Y_2, P_2', R_3, R_3, R_3', R_3', G_1', B_1')$, $\mathscr{P}_6 = (Y_1, Y_1, G_1, G_1, B_1, B_1, B_2', P_3)$, have respectively four, five, and six o.c. solutions. The chart shows that puzzle $\mathscr{P}_7 = (Y_2, Y_2, R_3, R_3, R_3', G_1', B_1', B_1')$ is compatible with seven o.c.'s as solutions, but the list shows that for all o.c.'s except Y_1, there is some corner at which no block of the puzzle fits. Therefore $\mathscr{P}_7$ has Y_1 as its only o.c. solution.

THEOREM 6. *There does not exist any puzzle which has more than six different o.c.'s as solutions.*

Editorial demand that the author cut the length of this paper forces omission of the proof, which requires exhaustion of twenty-four cases, and is seven pages in length. The author will supply a copy of the seven pages on request. Perhaps a reader will be able to construct a more elegant proof.

Reference

1. S. J. Kahan, Eight blocks to madness, a logical solution, this MAGAZINE, 45 (1972) 57–72.

ANOTHER EUCLIDEAN GEOMETRY

THOMAS P. DENCE, Colorado State University

Geometry instructors usually attempt at some point to explain geometry from a more abstract point of view. Rudimentary details of other geometries are quite commonly introduced. Students frequently enjoy such new "facts" as the existence of a counterexample to Euclid's Parallel Postulate, or of the existence of triangles which contain two right angles. This paper presents another geometry which may be used for the same purposes. In addition, a number of questions are raised which may be used as the basis of research projects for appropriate students, whether they be studying geometry, analysis, topology or the like.

The basic space. The setting for this geometry is a metric space $M = (S, d)$ where S is the set of elements and $d: S \times S \to R$, R = reals, is a distance function. This function must satisfy the following requirements:

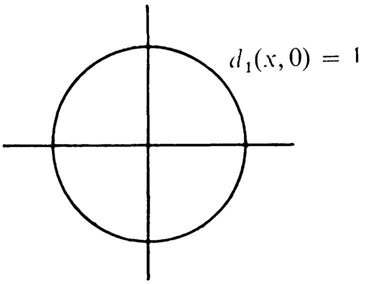
(a) $d(x,y) \geqq 0$ for x and y in S, and $d(x,y) = 0$ iff $x = y$

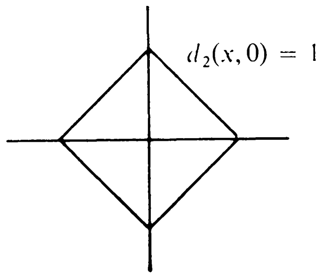
(b) $d(x,y) = d(y,x)$

(c) $d(x,z) \leqq d(x,y) + d(y,z)$.

The real Euclidean plane furnishes an example of a metric space in which $S = R^2$ and $d_1(x,y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Another example would be if $S = R^2$ and $d_2(x,y) = |x_1 - x_2| + |y_1 - y_2|$.

It is always helpful to view metrics by means of a diagram. So in the above two metrics, if we set $y = 0$ in both cases and consider $d(x,y) = 1$, then we get the following two diagrams:





Presently though, let $S = Q$, the set of rationals, and we let $d: Q \times Q \to R$ as follows [**1**]:

For each $x \in Q$, $x \neq 0$, let us represent x as $2^k(a/b)$ where k, a, and b are integers and 2 does not divide either a or b. This condition insures the uniqueness of the exponent k. For example, if $x = 10/17$ then $x = 2(5/17)$, or $x = 3/16 = 2^{-4}(3/1)$.

Let $|\ \ |_2 : Q \to R$ be defined by

$$|x|_2 = (1/2)^k \text{ and } |0|_2 = 0.$$

Thus $|10/17|_2 = 1/2$ and $|3/16|_2 = 16$. This function satisfies

(a) $|x|_2 \geqq 0$ for all x and $|x|_2 = 0$ iff $x = 0$,

(b) $|xy|_2 = |x|_2 \cdot |y|_2 = |yx|_2$,

(c) $|x + y|_2 \leqq |x|_2 + |y|_2$.

The function $|\ \ |_2$ somewhat resembles a norm on the space Q, but it fails to factor out scalars. To define a distance on a normed space, one usually investigates the norm of a difference. Thus we let $d: Q \times Q \to R$ be $d(x,y) = |x - y|_2$. For example, $d(1/2, 1/3) = |1/6|_2 = 2$. This function d satisfies the three requirements for a distance function. But in addition, it satisfies the inequality

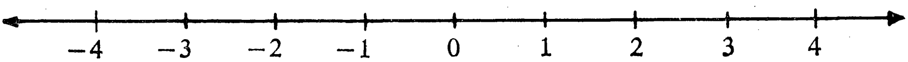
$$d(x,z) \leqq \max\{d(x,y), d(y,z)\}$$

called the *ultrametric inequality* [**1**].

Geometries based on such a metric space will be called ultrametric geometries. The remainder of this paper will deal with ultrametric geometries.

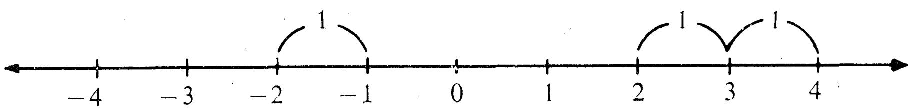
The definition of d could have been generalized by using $x = p^k(a/b)$ and $|x|_p = c^k$ for any prime p and any real number c such that $0 < c < 1$. Presently though, we will work with the particular function described above.

Some geometric properties. Let us represent Q by the number line



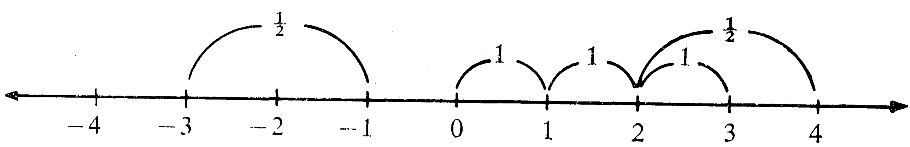
in which the usual ordering ($p < q$) holds true. Some of the properties of the distance function d are:

(1) If n is an integer, then $d(n, n+1) = |-1|_2 = 1$.



(Note: this is one of the few results that agree with Euclidean geometry.)

(2) If n is an integer, then $d(n, n+2) = 1/2$.



(3) If n is an integer, then $d(n, n+3) = 1$, and so on.

These metric properties show that our ultrametric geometry will certainly yield different-than-usual results. We now introduce some familiar geometric concepts.

(4) Let $S(x,r)$ denote the *sphere* of radius r with center at x, defined by

$$S(x,r) = \{y : d(x,y) = r\}.$$

In particular, consider the unit sphere centered at $x = 0$, denoted by $S(0,1)$. Then

$$S(0,1) = \pm[\{1,3,5,\cdots\} \cup \{1/3, 3/3, 5/3, \cdots\} \cup \{1/5, 3/5, \cdots\} \cup \cdots].$$

In other words, $S(0, 1)$ is actually the set of all rational combinations of odd integers.

(5) Let $\overline{AB}$ denote the *line segment* joining points A and B, defined as coinciding with the usual notion of line segment in the plane. Points A and B will be called the *endpoints* of the segment. Along with the concept of a line segment is that of the midpoint of the segment. Three fairly intuitive and seemingly equivalent midpoint definitions, which produce striking differences in results, are:

DEFINITION. I. *M is the midpoint of $\overline{AB}$ iff $A < M$ and $d(A, M) = (1/2)d(A, B)$.*

DEFINITION II. *M is the midpoint of $\overline{AB}$ iff $M < B$ and $d(B, M) = (1/2)d(A, B)$.*

DEFINITION III. *M is the midpoint of $\overline{AB}$ iff $d(A, M) = d(M, B) = (1/2)d(A, B)$.*

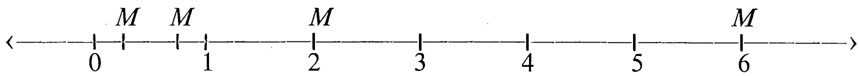
Then we let $\mathscr{M}$ denote the set of all midpoints of $\overline{AB}$. In particular, let us consider the segment with endpoints $A = 0$ and $B = 1$. According to the first definition, the midpoint M is that point such that

(a) $0 < M$ and $d(0, M) = (1/2)d(0, 1) = 1/2$.

Thus $M = 2, 6, 10, 14, \cdots; 2/3, 2/5, 2/7, \cdots$

and $\mathscr{M}_1 = \{x \colon x = 2 \cdot (a/b)$ where a and b are integers not divisible by $2\}$.

Hence we have a countably infinite number of midpoints.



According to the second definition we find

(b) $M < 1$ and $d(M, 1) = 1/2$.

Thus $M = -1, -5, \cdots; 1/3, 3/5, 7/9, \cdots$; and

$$\mathscr{M}_2 = \{x \colon x = 1 - 2(a/b), \text{ where } a \text{ and } b \text{ are integers not divisible by } 2\}.$$

Again we have a countably infinite number of midpoints, and yet $\mathscr{M}_1$ and $\mathscr{M}_2$ are disjoint. In fact, some of the midpoints, like -1 and -5, fall outside of the segment, as in case (a). But this could be eliminated by altering the definition of midpoint.

According to the third definition we have

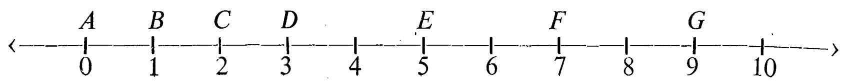
(c) $d(0, M) = d(M, 1) = 1/2$, and no M satisfies these equations simultaneously!

Thus $\mathscr{M}_3 = \varnothing$.

The choice of definition of segment midpoint has a great bearing on one's future use of such other geometric terms as "equidistant," "bisector of a segment," and "betweenness."

(6) Two segments $\overline{AB}$ and $\overline{CD}$ are said to be *equal* (the word "congruent" is sometimes preferred) if $d(A, B) = d(C, D)$. Thus the following segments are all equal:

$$\overline{AB} = \overline{BC} = \overline{CD} = \overline{AD} = \overline{AE} = \overline{AF} = \overline{CE} = \overline{CF} = \overline{CG}$$



(7) Let x be in Q. Then there exists a countable infinity of y in Q such that
(a) $|x-y|$ is small and $|x-y|_2$ is small ($|\ |$ denotes ordinary absolute value),
(b) $|x-y|$ is small and $|x-y|_2$ is not small,
(c) $|x-y|$ is not small and $|x-y|_2$ is small,
(d) $|x-y|$ is not small and $|x-y|_2$ is not small.

Example of (b): Let $x = 1$ and let $y \in \{1/2, 3/4, 7/8, 15/16, \cdots\}$. There are an infinite number of these y that are as close to $x = 1$ in ordinary Euclidean 1-space as we wish; i.e., $|y-1|$ is small, but

$$\begin{aligned} d(1,y) &= d(1,1/2) = 2 \\ &= d(1,3/4) = 4 \\ &= d(1,7/8) = 8 \\ &\ \ \vdots \qquad\quad \vdots \end{aligned}$$

is certainly not small.

(8) For a sequence of rationals $a_1, a_2, a_3, \cdots$ we define this sequence as being *Cauchy* if given $\varepsilon > 0$ there exists an integer N such that $m, n > N$ implies $d(a_n, a_m) < \varepsilon$ [1], and is *semi-Cauchy* if $d(a_n, a_{n+1}) < \varepsilon$. Furthermore the sequence (a_n) has a *limit*, A, denoted $a_n \to A$, if $d(A, a_n) \to 0$ as $n \to \infty$. We say (a_n) diverges to ∞, or $a_n \to \infty$, if given any number $K > 0$ there exists an integer N such that for some $m, n > N$ we have $d(a_n, a_m) > K$. Then we have

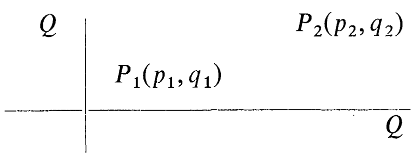
(a) $2, 2^2, 2^3, \cdots \to 0$,
(b) $2 \cdot r,\ 2^2 \cdot r,\ 2^3 \cdot r, \cdots \to 0$, r is rational,
(c) $1,\ 1/2,\ 1/3,\ 1/4, \cdots \to \infty$,
(d) $1,\ 3,\ 7,\ 15,\ 31, \cdots \to -1$.

The study of sequences in this ultrametric geometry provides one with many questions. In concluding this section, the following questions can be posed: How many nonconstant sequences converge and what is the nature of their limits? Is every semi-Cauchy sequence also Cauchy? Does every Cauchy sequence have a limit, or is the space complete?

Extension to $Q \times Q$. Much more can be said about the application of the function d along the one-dimensional axis, but the instructor of geometry is just as interested, if not more so, in dimensions greater than one. So let us consider, in particular, the space $Q \times Q$ and see if we can extend the function d to give a distance function on $Q \times Q$ and also satisfy the ultrametric inequality.

One possible extension of d would be to define

$$D: Q^2 \times Q^2 \to R \text{ by } D(P_1, P_2) = d(p_1, p_2) + d(q_1, q_2).$$



We then have $D \equiv d$ on the Q-axis. To verify that D is actually a distance function, we note

(a) $D(P_1, P_2) \geqq 0$ for all P_i,
(b) $D(P_1, P_2) = 0$ iff $P_1 = P_2$,
(c) $D(P_1, P_2) = D(P_2, P_1)$,
(d) $D(P_1, P_3) \leqq D(P_1, P_2) + D(P_2, P_3)$.

But the question that remains is, "Does D satisfy the ultrametric inequality?" Is $D(P_1, P_3) \leqq \max\{D(P_1, P_2), D(P_2, P_3)\}$? Or, is

$$d(p_1, p_3) + d(q_1, q_3) \leqq \max\{d(p_1, p_2) + d(q_1, q_2), d(p_2, p_3) + d(q_2, q_3)\}?$$

Now we know

$$d(p_1, p_3) \leqq \max\{d(p_1, p_2), d(p_2, p_3)\} \text{ and } d(q_1, q_3) \leqq \max\{d(q_1, q_2), d(q_2, q_3)\}.$$

Thus we have

$$d(p_1, p_3) + d(q_1, q_3) \leqq \max\{d(p_1, p_2), d(p_2, p_3)\} + \max\{d(q_1, q_2), d(q_2, q_3)\}.$$

Obviously this does not satisfy the ultrametric inequality, except for a "sometimes" reply.

One could continue a study of $Q \times Q$ with the function D but I think it would be more profitable to work with a function which does satisfy the ultrametric inequality. Unfortunately, the author could not exhibit a "good one." The metric

$$D^*(P_1, P_2) = \begin{cases} 0, & \text{if } P_1 = P_2 \\ 1, & \text{if } P_1 \neq P_2 \end{cases}$$

satisfies the ultrametric inequality, except it is rather trivial. On the other hand the function

$$D^*(P_1, P_2) = d(p_1, p_2) \text{ where } P_1 = (p_1, q_1), P_2 = (p_2, q_2)$$

satisfies the ultrametric inequality, but is not a distance function because $D^*(P_1, P_2) = 0$ fails to imply $P_1 = P_2$. Such a function is called a pseudo-metric.

To find a "good" function, as desired above, shall be left as one of those "left to the reader" exercises. The only choice left is to deal in general terms without any particular function on hand.

Extension to the abstract. Suppose we have an arbitrary metric space $M = (X, d)$ in which d satisfies the ultrametric inequality. Some results follow:

(1) In discussing triangles in M, where a triangle could be defined as that which is determined by the union of 3 line segments with 3 distinct endpoints x, y, z, we have the following: All triangles in (X, d) are at least isosceles [**1**].

Proof. Assume $d(x, y) \neq d(y, z)$.
Case 1: $d(x, y) > d(y, z)$,

$$\therefore d(x, z) \leqq \max\{d(x, y), d(y, z)\} = d(x, y),$$

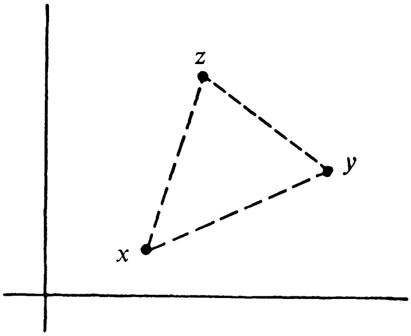
and

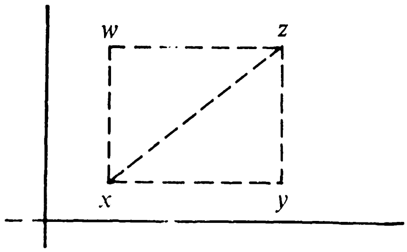
$$d(x,y) \leqq \max\{d(x,z), d(z,y)\} = d(x,z),$$

$$\therefore \; d(x,z) = d(x,y).$$

Case 2: $d(x,y) < d(y,z)$.
Same procedure as above.

(2) Squares exist, but their diagonals are of length no greater than the sides.





One could have some "fun" in discussing areas and congruences of polygons. As is usually the case, we define an *open ball* $B(x;r)$ in M to consist of all points y in X whose distance from x is strictly less than r. A *closed ball* $B'(x;r)$ is the same except it allows $d(x,y) = r$. Then we have [2]

(3) $B(x;r)$ [$B'(x;r)$] is both an open set and a closed set.

(4) For any $y \in B(x;r)$, we have $B(y;r) = B(x;r)$.

Proof. Let $z \in B(x;r)$; thus $d(x,z) < r$. But $d(y,z) \leqq \max\{d(y,x), d(x,z)\} < r$.

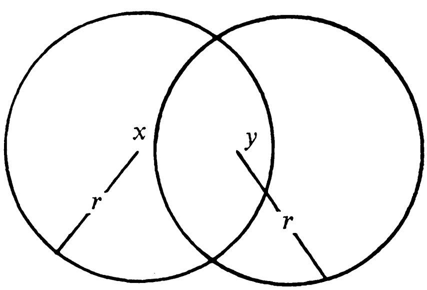
$$\therefore \; z \in B(y;r) \rightarrow B(x;r) \subseteq B(y;r).$$

Let $z \in B(y;r)$ and similarly we get

$$B(y;r) \subseteq B(x;r).$$

Hence $B(x;r) = B(y;r)$.

Thus every point inside an open ball is a center.



(5) If two open balls in M have a common point, then one of them is contained in the other. And finally we have

(6) The distance between two distinct open balls of radius r, contained in a closed ball of radius r, is equal to r.

Proof. Let $B'(x; r)$ be the closed ball, and $B_1(x_1; r)$, $B_2(x_2; r)$ be the open balls. Then $d(x_1, x_2) \geqq r$ and $d(x, x_i) \leqq r$. Let $a \in B_2$ and $b \in B_1$.

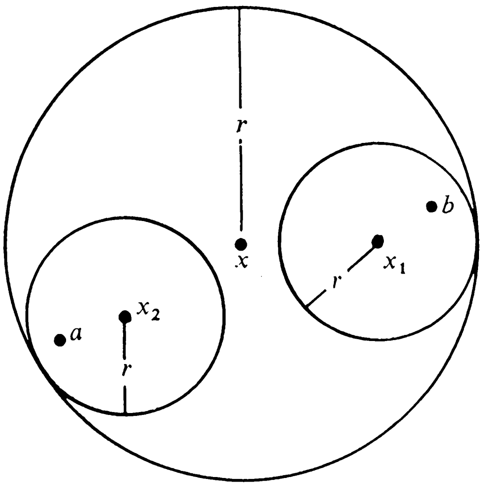
$$\therefore\ d(a,b) \leqq \max\{d(a,x), d(x,b)\} \leqq r.$$

Also $a \notin B_1 \rightarrow d(a, x_1) \geqq r$.

$$\therefore\ r \leqq d(a,x_1) \leqq \max\{d(a,b), d(b,x_1)\} = d(a,b).$$

$$\therefore\ r = d(a,b),$$

$\therefore$ the two balls are r units apart!



because otherwise we would have $r \leqq d(b, x_1) < r$.

To many, at least those that are not mathematically oriented, this last statement would seem contradictory to common sense, yet it is true in the geometry so defined. An example of another assertion, seemingly opposed to common sense, and perhaps more astonishing, is furnished by the Banach-Tarski paradox [3]. This paradox involves bounded, nonempty interior sets in any real Euclidean space of dimension greater than two. In brief it says, for example, that an orange may be decomposed into a finite number of mutually disjoint "pieces," these pieces then rearranged so as to form the earth, and vice versa! So in this sense, it is possible to hold the whole wide world in your hands!

The method of presenting this material to a class is left to the instructor. Whether he chooses to mention the results just casually or to present a more in-depth study, the student reaction could quite possibly consist of such disbelieving remarks as "sure," "show us," "impossible" or "No Way!"

There is, and rightly so, something to be said in higher mathematics about the previously defined functions $| \ |_p$ and $d(x,y) = |x - y|_p$. The function $| \ |_p$ is an example of a *valuation* on the field of rationals, and is essentially the only nontrivial valuation which satisfies the additional property of $|a + b| \leqq \max(|a|, |b|)$, hence it is called a nonarchimedean valuation. A valuation is a function that acts much like the absolute-value function.

A substantial portion of analysis and algebra is devoted to studying both functions of this type and fields with a valuation defined on them. For example, given a field F with a valuation $| \ |$ and distance d, one can discuss sequences, convergence and completeness of the field. Any field F with a valuation $| \ |$ can be "extended" to a complete field $\hat{F}$ [**1**]. The completion of the rationals with respect to $| \ |_p$ gives the complete field of the p-adic numbers. It is in these fields that we produce such equations as

$$-1 = 4 + 4\cdot 5 + 4\cdot 5^2 + 4\cdot 5^3 + 4\cdot 5^4 + \ldots$$

$$\sqrt{7} = 1 + 1\cdot 3 + 1\cdot 3^2 + 0\cdot 3^3 + 2\cdot 3^4 + \ldots.$$

References

1. George Bachman, Introduction to p-Adic Numbers and Valuation Theory, Academic Press, New York, 1964, pp. 1–4.

2. J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960, p. 38.

3. L. M. Blumenthal, A paradox, a paradox, a most ingenious paradox, Amer. Math. Monthly, 47 (1940) 346–353.

ISOLATION, A GAME ON A GRAPH

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1. Preliminaries. Before discussing the game, we define some graph theory terms. (See any or all of [**1**]–[**4**] for further graph theory investigation.)

A *graph* consists of a finite nonempty set $V = V(G)$ of *vertices* together with a prescribed set $E = E(G)$ of distinct unordered pairs of distinct vertices called *edges*. Two vertices making a pair in $E(G)$ are said to be *adjacent*. It is customary to draw a diagram of a graph where the vertices are represented as points and the edges as line segments joining the points. Thus, in Figure 1, u, v, w, z and t are vertices. The pair uv is in $E(G)$, while uw is not. The vertex z is adjacent to the vertices u and w. We will refer to such diagrams as the graphs themselves.

A graph is *connected* if one can trace along edges and vertices from any vertex to any other. The graph in Figure 1 is not connected since one cannot trace from w to t, for example.

The *degree* of a vertex v, $d(v)$, is the number of edges which meet v. Equivalently, it is the number of vertices with which the given vertex is adjacent. An *isolated vertex* is one of degree zero. The maximum degree of any vertex in a graph with n vertices is $n - 1$. In Figure 1, t is an isolated vertex while all other vertices have degree 2.

There is, and rightly so, something to be said in higher mathematics about the previously defined functions $| \ |_p$ and $d(x,y) = |x - y|_p$. The function $| \ |_p$ is an example of a *valuation* on the field of rationals, and is essentially the only nontrivial valuation which satisfies the additional property of $|a + b| \leqq \max(|a|,|b|)$, hence it is called a nonarchimedean valuation. A valuation is a function that acts much like the absolute-value function.

A substantial portion of analysis and algebra is devoted to studying both functions of this type and fields with a valuation defined on them. For example, given a field F with a valuation $| \ |$ and distance d, one can discuss sequences, convergence and completeness of the field. Any field F with a valuation $| \ |$ can be "extended" to a complete field $\hat{F}$ [**1**]. The completion of the rationals with respect to $| \ |_p$ gives the complete field of the p-adic numbers. It is in these fields that we produce such equations as

$$-1 = 4 + 4\cdot 5 + 4\cdot 5^2 + 4\cdot 5^3 + 4\cdot 5^4 + \dots$$

$$\sqrt{7} = 1 + 1\cdot 3 + 1\cdot 3^2 + 0\cdot 3^3 + 2\cdot 3^4 + \dots.$$

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ISOLATION, A GAME ON A GRAPH

R. D. RINGEISEN, Colgate University

1. Preliminaries. Before discussing the game, we define some graph theory terms. (See any or all of [**1**]–[**4**] for further graph theory investigation.)

A *graph* consists of a finite nonempty set $V = V(G)$ of *vertices* together with a prescribed set $E = E(G)$ of distinct unordered pairs of distinct vertices called *edges*. Two vertices making a pair in $E(G)$ are said to be *adjacent*. It is customary to draw a diagram of a graph where the vertices are represented as points and the edges as line segments joining the points. Thus, in Figure 1, u, v, w, z and t are vertices. The pair uv is in $E(G)$, while uw is not. The vertex z is adjacent to the vertices u and w. We will refer to such diagrams as the graphs themselves.

A graph is *connected* if one can trace along edges and vertices from any vertex to any other. The graph in Figure 1 is not connected since one cannot trace from w to t, for example.

The *degree* of a vertex v, $d(v)$, is the number of edges which meet v. Equivalently, it is the number of vertices with which the given vertex is adjacent. An *isolated vertex* is one of degree zero. The maximum degree of any vertex in a graph with n vertices is $n - 1$. In Figure 1, t is an isolated vertex while all other vertices have degree 2.

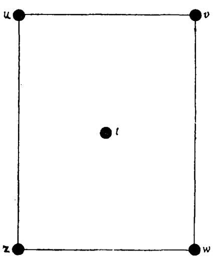


FIG. 1. A Graph Diagram

In [5], the authors define the operation of switching a graph. By *switching a graph G at a vertex v*, we mean creating a new graph G' on the same number of vertices by removing all edges of G meeting v, adding edges from v to any vertex in G that is not adjacent to v, and leaving all other edges unchanged. In Figure 2 we switch G at vertex w to get G'. We then switch G' at vertex z to get G''. Notice that the switch from G' to G'' resulted in z having degree zero. The vertex z was adjacent to every other vertex in G'. Hence, the switch removed all possible edges from z. (w shared the fate of z, why?)

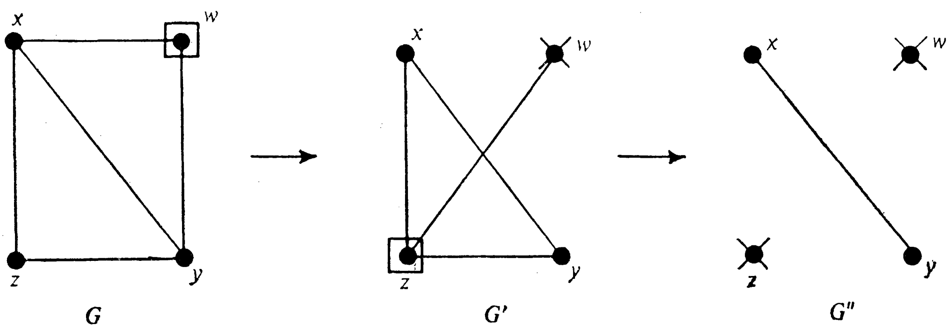


FIG. 2. Switching

Switching with respect to a set of vertices is defined to be switching with respect to each vertex in the set, one after another. (One can show that the order makes no difference.) Switching at a vertex v changes the degree of v from $d(v)$ to $(n-1) - d(v)$.

2. Isolation. Isolation is a game for two players, whom we call X and Y. It is played as follows:

A. X and Y agree as to the number of vertices, n, the playing graph will have.

B. Player X chooses a connected graph on n vertices.

C. Player Y chooses a vertex of the graph and switches with respect to that vertex.

D. Player X now chooses a vertex from among the vertices Y did not use and switches the new graph with respect to it.

The players alternately switch the graphs, under the stipulation that once a vertex is used it cannot be used again by either player. The first person whose switch results in at least one isolated vertex wins the game.

In Figure 3 a complete game of isolation is played with player X winning.

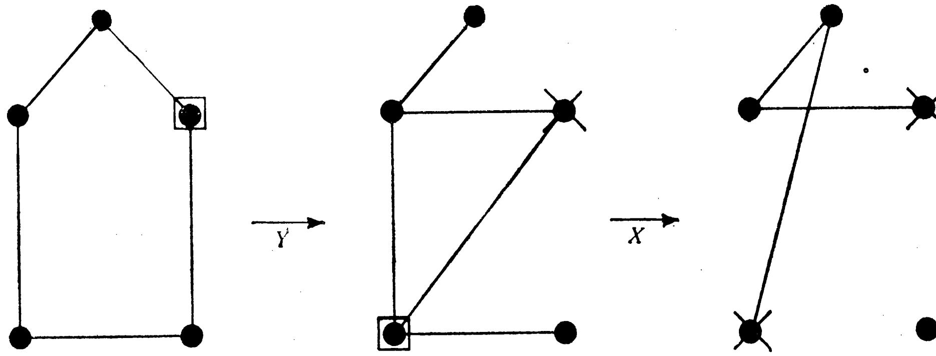


FIG. 3. A Game of Isolation

The box denotes the vertex used by the player involved to obtain the next graph. An × indicates a vertex has already been used.

We now turn to discussing some questions concerning switching which make isolation feasible.

3. Some relevant properties of switching. The reader can easily convince himself of the following theorem:

THEOREM 1. *Let $G * H$ mean that the graph H may be obtained from the graph G by a finite number of single vertex switchings. Then $*$ is an equivalence relation on the set of graphs with n vertices, for all n.*

The following theorem justifies the existence of isolation:

THEOREM 2. *In any switching equivalence class there is at least one graph having an isolated vertex.*

Proof. Choose any graph G from the given equivalence class C. If G has an isolated vertex, we're finished. If not, let v be any vertex of G and let $v_1, v_2, \cdots, v_{d(v)}$ be the vertices which are adjacent to v. It is clear that switching with respect to the set $\mathscr{S} = \{v_1, v_2, \cdots, v_{d(v)}\}$ results in v becoming an isolated vertex.

Theorem 2 seems to show that any connected graph can be chosen to begin a game of isolation. However, the theorem does not consider the certain prospect that one or the other of the players may prevent the isolated vertex from ever appearing. So, although the theorem does theoretically justify the game, it does not assure us that every game is winnable. We thus discuss winnability in isolation.

4. Some remarks and theorems. In what follows, X will always mean the player who chooses the graph while Y will indicate the other player (who switches first).

DEFINITION. *A player is using perfect end game play which we call perfect play for brevity, if whenever a vertex of the graph before him could be used to* *switch to an isolated vertex graph, he will make such a switch. Furthermore, if he cannot win but there is a switch which will result in a graph that his opponent cannot use to win, he will make that switch. Perfect play also assumes that X will not choose a graph for which Y can win on his first move. A game has perfect play if both players use perfect end game play.*

DEFINITION. *A graph is P favorable if when isolation is perfectly played, P always wins (P = X or Y).*

A graph is *winnable* if, under perfect play, someone must win. We leave the proof of the following theorem to the reader:

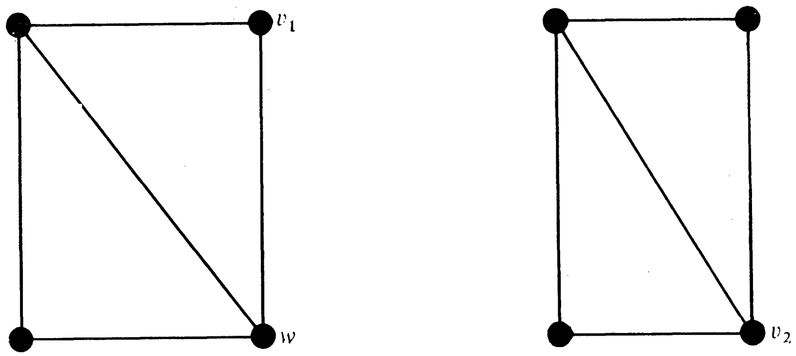


FIG. 4. Vertices of Degree One and n–1

THEOREM 3 (See Figure 4). *Let G be a graph with n vertices upon which a player P is to act during a perfectly played game of isolation. If G has either a vertex of degree one or a vertex of degree* $(n-1)$, *P will win the game.*

Let K_n be the complete graph on n vertices. That is, K_n has any two of its vertices adjacent. Let C_n, the cycle graph on n vertices, be the connected graph on n vertices so that every vertex has degree two. (See Figure 5.)

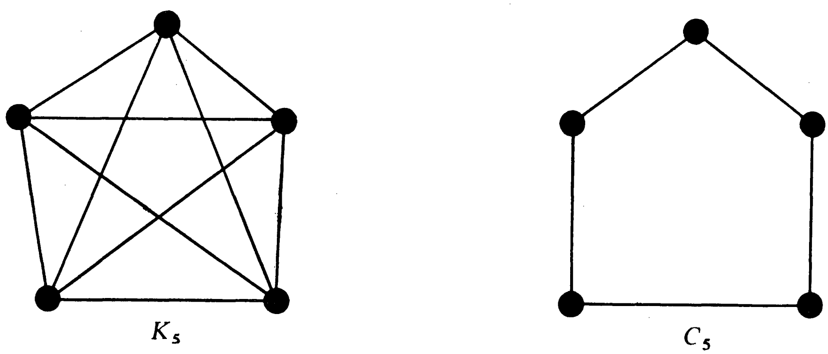


FIG. 5. The Complete Graph and the Cycle Graph

The reader can prove the following corollaries to Theorem 3:

COROLLARY 1. *K_n is Y favorable.*

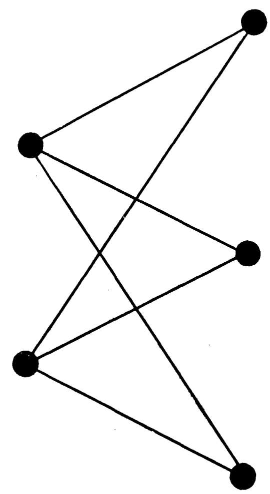
COROLLARY 2. *C_n is X favorable.*

A player may use Theorem 3 to determine his move. Actually, Theorem 3 completely determines when a player may win. In other words, it tells him what to look for and what to avoid giving!

THEOREM 4. *A player may win immediately only if he is given a graph which has either a degree 1 or a degree $n-1$ vertex.*

Proof. A vertex v becomes isolated by switching either at it or at another vertex. Switching at another vertex can remove at most one adjacency from v. Switching at v can make v isolated only if all of its adjacencies were present.

We now examine whether all graphs are winnable. The reader may convince himself that the graph in Figure 6 is nonwinnable. Are there nonwinnable graphs on any number of vertices? Before stating the theorem which leads to an answer, we define a *complete bipartite graph*, $K_{m,n}$. Let V_m *and* V_n be sets of m and n distinct vertices, respectively. $K_{m,n}$ is formed when an edge is drawn from each vertex of V_m to each vertex of V_n. No edges are drawn among vertices of V_m, nor among those of V_n. (See Figure 6.) Let $\overline{K_t}$ denote the graph on t vertices that has no edges.



$K_{2,3}$

FIG. 6. The Complete Bipartite Graph

THEOREM 5. *Let C be the switching equivalence class on n vertices which contains $\overline{K_n}$. Then $\overline{K_n}$ is the only disconnected graph in C. Moreover, each graph of C which is not $\overline{K_n}$ is a complete bipartite graph, $K_{p,n-p}$.*

Proof. It is clear that one may obtain all members of an equivalence class by choosing any member and switching with respect to each subset of this member's vertices. We choose $\overline{K_n}$. Switching $\overline{K_n}$ at any vertex yields $K_{1,n-1}$, a complete bipartite graph.

When a complete bipartite graph is switched at a vertex (other than that vertex of $K_{1,n-1}$ which has degree $n-1$) another complete bipartite graph results. To see this, let v be a vertex of V_p in the graph $K_{p,n-p}$, $p \neq 1$. Switching at v results in $K_{p-1,n-p+1}$ since v is then adjacent to nothing in V_{n-p} and to each member of V_p.

Since switching with respect to a set is sequential one point switching, the theorem is proven.

Theorem 5 may be used to show that there are nonwinnable graphs on any set of vertices with five or more members.

THEOREM 6. *For every integer $n \geqq 5$, $K_{p,n-p}$, $2 \leqq p \leqq n-2$, is a nonwinnable graph.*

Proof. By Theorem 5, only $\overline{K_n}$ is an obtainable isolated vertex graph. We note that to switch from any complete bipartite graph to $\overline{K_n}$, one must first obtain $K_{1,n-1}$. Similarly, before obtaining $K_{1,n-1}$ one must obtain $K_{2,n-2}$. Hence, if whenever a player is faced with $K_{2,n-2}$ he can avoid switching to $K_{1,n-1}$, the game will never be won. We establish the following claim:

CLAIM. *If, during a perfectly played game of isolation, a player is faced with $K_{2,n-2}$, either all vertices have been used, or there is at least one vertex in V_{n-2} which has yet to be used.*

Proof of claim. Assume that all vertices in V_{n-2} have been used. Then no two vertices in V_{n-2} were adjacent in the original game graph. (Switching with respect to a set of two or more vertices leaves adjacencies among vertices of the set unchanged.) Thus, by Theorem 5, the original graph was either $K_{n-2,2}$ or $K_{n-1,1}$. Assuming perfect play outlaws $K_{n-1,1}$. If one switches $K_{n-2,2}$ by each of the $n-2$ vertices of degree two, while using neither of the two vertices of degree $n-2$, he obtains $\overline{K_n}$, a contradiction of hypothesis. So, at least one vertex of degree $(n-2)$ in the $K_{2,n-2}$ graph facing the player must have been used.

Suppose exactly one vertex, v, of V_2 has been used. Then, in the original graph, v was adjacent to none of the vertices in V_{n-2} and was adjacent to the other vertex in V_2. Hence, the original graph was $K_{n-1,1}$, a contradiction of perfect play.

Hence, all vertices in the $K_{n-2,2}$ facing the player have been used. This proves the claim.

By the claim, when $K_{2,n-2}$ appears either all vertices have been used and the game thus ends, or the player may choose one of the $n-2$ vertices of degree 2. The resultant graph avoids $K_{1,n-1}$. Thus the game is nonwinnable.

An interesting open question is to characterize nonwinnable graphs. Are those described in Theorem 6 the only ones? We investigate the question no further here. In the next section, we introduce a new method for playing isolation.

5. The games and matrices. In [**5**], the authors introduce and use the following adjacency matrix. (It differs from those found in [**1**]–[**4**].) Let G be a graph and let its vertices be numbered $1, 2, \cdots, n$. The $+, -$ *adjacency matrix of* G is defined by:

$$(A)_{i,j} = \begin{cases} -1 \text{ if } i \text{ and } j \text{ are adjacent} \\ +1 \text{ if } i \text{ and } j \text{ are nonadjacent} \\ \ \ 0 \text{ if } i = j. \end{cases}$$

When a graph G with adjacency matrix A is switched at vertex i, the adjacency matrix A' for the new graph can be obtained by multiplying the ith row and ith column of A by -1. Equivalently, we could pre- and post-multiply A by a diagonal matrix whose nonzero entries are all $+1$ except for the i, i entry, which is -1. Switching with respect to a set $\mathcal{S}$ could be done by pre- and post-multiplying by a diagonal matrix so that the entries corresponding to elements of $\mathcal{S}$ are -1, those not in $\mathcal{S}$ are $+1$. The authors of [**5**] use eigenvalues and other matrix properties to analyze the switching operation. The reader might find it instructional to examine the adjacency matrix of $K_{m,n}$ and then prove Theorem 5 using matrix multiplication.

We mention matrices here only so that anyone who would rather matrix multiply than switch is aware of that option. We encourage you to do some matrix switching.

6. Conclusion. The switching operation is interesting and has many open questions on its own. We here confined the discussion to its gaming implications in the hope that further investigations of isolation might ensue.

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GEOMETRIC APPROXIMATIONS FOR *e*

J. D. BUCKHOLTZ, University of Kentucky

1. Rectangles. The definition of e by means of the definite integral [**1**, p. 231]

$$1 = \int_1^e \frac{1}{x} dx$$

and the interpretation of this integral as an area suggest a geometric method for obtaining numerical estimates for e. Approximate the area in question with rectangular strips in the usual fashion, temporarily ignoring the fact that the vertical line which bounds the area on the right is not known. If the rectangles have total area 1 and their tops lie above the curve $y = 1/x$, the rectangles will not reach as far as e. If the rectangles have total area 1 and are drawn under the curve $y = 1/x$, they will extend past e. Using two rectangles, each of area 1/2, it is easy to check that the bases of the rectangles will be $[1, 3/2]$ and $[3/2, 9/4]$ in the first case and $[1, 2]$ and $[2, 4]$ in the second case. Therefore $9/4 < e < 4$.

To carry out the argument in general, we suppose that n is a fixed integer greater than 1 and that $\{s_k\}_{k=0}^n$ and $\{t_k\}_{k=0}^n$ are increasing sequences such that $s_0 = t_0 = 1$.

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From area considerations (or the definition of an integral) we have

$$(1)\qquad \int_1^{s_n} \frac{1}{x}dx < \sum_{k=1}^{n} \frac{s_k - s_{k-1}}{s_{k-1}} \text{ and} \int_1^{t_n} \frac{1}{x}dx > \sum_{k=1}^{n} \frac{t_k - t_{k-1}}{t_k}.$$

If our sequences $\{s_k\}$ and $\{t_k\}$ satisfy

$$(2)\qquad \sum_{k=1}^{n} \frac{s_k - s_{k-1}}{s_{k-1}} = \sum_{k=1}^{n} \frac{t_k - t_{k-1}}{t_k} = 1,$$

then (1) will imply that $s_n < e < t_n$.

How shall $\{s_k\}$ and $\{t_k\}$ be chosen? The obvious choice is to *make the terms in* (2) *equal*, i.e., set

$$(3)\qquad \frac{s_k - s_{k-1}}{s_{k-1}} = \frac{t_k - t_{k-1}}{t_k} = \frac{1}{n}.$$

Solving (3) for s_k and t_k we obtain

$$s_k = \frac{n+1}{n}s_{k-1} \text{ and } t_k = \frac{n}{n-1}t_{k-1}.$$

Since $s_0 = t_0 = 1$ we have

$$s_k = \left(\frac{n+1}{n}\right)^k = \left(1 + \frac{1}{n}\right)^k \text{ and } t_k = \left(\frac{n}{n-1}\right)^k = \left(1 + \frac{1}{n-1}\right)^k.$$

Therefore our "geometric" inequality $s_n < e < t_n$ asserts that

$$(4)\qquad \left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n-1}\right)^n, \qquad n = 2, 3, 4, \cdots.$$

In the right half of (4), replace n by $n + 1$. This yields

$$e < \left(1 + \frac{1}{n}\right)^n\left(1 + \frac{1}{n}\right) < \left(1 + \frac{1}{n}\right)^n + \frac{e}{n}.$$

Combining this inequality with (4), we obtain the well known result that

$$0 < e - \left(1 + \frac{1}{n}\right)^n < \frac{4}{n}, \qquad n = 1, 2, 3, \cdots.$$

Can our choice of the sequences $\{s_k\}$ and $\{t_k\}$ be improved? The best choice would *maximize* s_n and *minimize* t_n, subject to the constraints (2). If we rewrite (2) in the simpler form

$$\sum_{k=1}^{n} \frac{s_k}{s_{k-1}} = n + 1 \text{ and } \sum_{k=1}^{n} \frac{t_{k-1}}{t_k} = n - 1,$$

and note that

$$s_n = \prod_{k=1}^{n} \frac{s_k}{s_{k-1}} \text{ and } t_n^{-1} = \prod_{k=1}^{n} \frac{t_{k-1}}{t_k},$$

our problem reduces to that of maximizing a product of factors whose sum remains fixed. But the solution to this problem is well known: the product is largest when the factors are equal (the geometric mean never exceeds the arithmetic mean) [**1**, p. 47]. Therefore, best results will be obtained by taking

$$\frac{s_k}{s_{k-1}} = \frac{n+1}{n} \text{ and } \frac{t_{k-1}}{t_k} = \frac{n-1}{n}, \qquad 1 \leqq k \leqq n.$$

This leads to

$$s_n = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \text{and } t_n = \left(\frac{n}{n-1}\right)^n = \left(1 + \frac{1}{n-1}\right)^n;$$

therefore our first choice is the best one.

2. Trapezoids. The graph of $y = 1/x, x > 0$, is convex; chords joining two points of the curve lie above the curve, and tangent lines lie below it. We can improve our estimates for e by replacing rectangles by trapezoids. As before, let $\{u_k\}_{k=0}^n$ and $\{v_k\}_{k=0}^n$ be increasing sequences such that $u_0 = v_0 = 1$. For $1 \leqq k \leqq n$, let U_k be the trapezoid with base $[u_{k-1}, u_k]$ and altitudes $1/u_{k-1}$ and $1/u_k$; let V_k be the trapezoid with base $[v_{k-1}, v_k]$ and top lying on the line tangent to $y = 1/x$ at $x = (v_{k-1} + v_k)/2$. (Note that the trapezoid V_k is not uniquely determined by its base $[v_{k-1}, v_k]$ and the requirement that its top be tangent to $y = 1/x$. Among all such trapezoids, V_k has the largest area.) The area under $y = 1/x$ between 1 and u_n is contained in the trapezoids U_k, and the area under $y = 1/x$ between 1 and v_n contains the trapezoids V_k. The inequality $u_n < e < v_n$ will result if we require that each set of trapezoids has total area 1. An easy calculation shows that this is equivalent to requiring that

(5)
$$\sum_{k=1}^{n} \frac{u_{k-1}^{-1} + u_k^{-1}}{2}(u_k - u_{k-1}) = 1$$

and

(6)
$$\sum_{k=1}^{n} \frac{v_k - v_{k-1}}{v_k + v_{k+1}} = \frac{1}{2}.$$

We shall attempt to maximize u_n subject to the constraints (5) and $1 \leqq u_{k-1} \leqq u_k$, $1 \leqq k \leqq n$, and to minimize v_n, subject to (6) and $1 \leqq v_{k-1} \leqq v_k$, $1 \leqq k \leqq n$. If we set $\alpha_k = u_k/u_{k-1}$ and $\beta_k = v_{k-1}/v_k$, our problem becomes that of maximizing

$$u_n = \prod_{k=1}^{n} \alpha_k \text{ and } v_n^{-1} = \prod_{k=1}^{n} \beta_k,$$

subject to the respective constraints

(5′)
$$\sum_{k=1}^{n} \frac{\alpha_k - \alpha_k^{-1}}{2} = 1, \ \alpha_k \geqq 1, \quad 1 \leqq k \leqq n,$$

and

$$\sum_{k=1}^{n} \frac{1-\beta_k}{1+\beta_k} = \frac{1}{2}, \ 0 < \beta_k \leqq 1, \ 1 \leqq k \leqq n. \tag{6'}$$

Can the problem be restated in simpler form? A possible approach is suggested by the summand in (5′), which slightly resembles a hyperbolic sine. If we write $\alpha_k = e^{x_k}$, then (5′) becomes

$$\sum_{k=1}^{n} \sinh x_k = 1, \quad x_k \geqq 0, \quad 1 \leqq k \leqq n, \tag{5''}$$

and the quantity we wish to maximize is

$$u_n = \prod_{k=1}^{n} \alpha_k = \exp\left\{\sum_{k=1}^{n} x_k\right\}.$$

Progress! The exponential function is increasing, so that maximizing u_n is equivalent to maximizing $\sum_{k=1}^{n} x_k$. Recall [**2**, p. 72] that a function f whose graph is convex satisfies

$$f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \leqq \frac{1}{n}\sum_{k=1}^{n} f(x_k), \tag{7}$$

and that the graph of $y = \sinh x, x \geqq 0$, *is* convex. Therefore

$$\sinh\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \leqq \frac{1}{n}\sum_{k=1}^{n} \sinh x_k = \frac{1}{n},$$

with equality holding if the x_k's are equal.

Success! The hyperbolic sine is increasing. Therefore

$$\frac{1}{n}\sum_{k=1}^{n} x_k \leqq \sinh^{-1}(1/n)$$

and

$$u_n = \exp\left\{\sum_{k=1}^{n} x_k\right\} \leqq \exp\{n \sinh^{-1}(1/n)\} = \left(\frac{\sqrt{n^2+1}+1}{n}\right)^n,$$

with equality holding if the x_k's are equal. Note that equal x_k's correspond to trapezoids of equal area.

We now apply the same procedure to maximize $v_n^{-1} = \prod_{k=1}^{n} \beta_k$, subject to (6′). Let $\beta_k = e^{y_k}$ and rewrite (6′) as

$$\sum_{k=1}^{n} \frac{1-e^{y_k}}{1+e^{y_k}} = -\sum_{k=1}^{n} \tanh(y_k/2) = \frac{1}{2}, \quad -\infty < y_k \leqq 0, \ 1 \leqq k \leqq n. \tag{6''}$$

The quantity we wish to maximize, subject to (6″), is

$$v_n^{-1} = \prod_{k=1}^{n} \beta_k = \exp\left\{\sum_{k=1}^{n} y_k\right\}.$$

Rewrite (6″) as

$$(6''') \qquad \sum_{k=1}^{n} \tanh(y_k/2) = -\tfrac{1}{2}, \qquad -\infty < y_k \leqq 0,\ 1 \leqq k \leqq n,$$

and observe that the hyperbolic tangent is increasing and convex on $(-\infty, 0]$. Repeating our previous argument, we obtain

$$\tanh\left(\frac{1}{n}\sum_{k=1}^{n} y_k/2\right) \leqq \frac{1}{n}\sum_{k=1}^{n} \tanh(y_k/2) = \frac{-1}{2n},$$

with equality holding if the y_k's are equal. Therefore

$$\frac{1}{n}\sum_{k=1}^{n} y_k/2 \leqq \tanh^{-1}\left(\frac{-1}{2n}\right),$$

and

$$v_n^{-1} = \exp\left\{\sum_{k=1}^{n} y_k\right\} \leqq \exp\left\{2n\tanh^{-1}\left(\frac{-1}{2n}\right)\right\},$$

so that

$$v_n \geqq \exp\left\{2n\tanh^{-1}\left(\frac{1}{2n}\right)\right\} = \left(\frac{2n+1}{2n-1}\right)^n,$$

with equality holding if the y_k's are equal. Again, equal y_k's correspond to trapezoids of equal area.

Combining our estimates, we see that

$$(8) \qquad \left(\frac{\sqrt{n^2+1}+1}{n}\right)^n < e < \left(\frac{2n+1}{2n-1}\right)^n, \qquad n = 1, 2, 3, \cdots$$

are the best estimates obtainable from trapezoidal approximations.

Why are convexity properties needed in the trapezoidal case but not in the rectangular case? Part of the answer lies in the fact that the rectangular estimates used only the fact that $y = 1/x, x > 0$, is decreasing, not that it is convex. A more satisfactory answer is that convexity *was* used in the rectangular case. The theorem of the means (geometric never exceeds arithmetic) is an immediate consequence of the convexity of the exponential function. If $\{p_k\}_{k=1}^n$ is a positive sequence, then exponential convexity implies that

$$\left\{\prod_{k=1}^{n} p_k\right\}^{1/n} = \exp\left\{\frac{1}{n}\sum_{k=1}^{n} \log p_k\right\}$$

$$\leqq \frac{1}{n}\sum_{k=1}^{n} \exp\{\log p_k\}$$

$$= \frac{1}{n}\sum_{k=1}^{n} p_k.$$

It seems unlikely that a shorter proof exists for the theorem of the means.

For strictly convex functions f, equality occurs in (7) if and only if the x_k's are equal [**2**, p. 74]. None of our results require change if we allow the sequences $\{s_k\}$, $\{t_k\}$, $\{u_k\}$, and $\{v_k\}$ to be nondecreasing rather than strictly increasing; this allows us to decrease the actual number of rectangles (trapezoids) without changing n. Together with our previous remark, this observation implies that the sequences

$$\left(1+\frac{1}{n}\right)^n \text{ and } \left(\frac{\sqrt{n^2+1}+1}{n}\right)^n, \qquad n=1,2,3,\cdots$$

are strictly increasing, and that the sequences

$$\left(1+\frac{1}{n}\right)^{n+1} \text{ and } \left(\frac{2n+1}{2n-1}\right)^n, \qquad n=1,2,3,\cdots$$

are strictly decreasing.

For large values of n, our trapezoidal approximations are significantly better than the rectangular approximations. Standard techniques yield the asymptotic estimates

$$\left(1+\frac{1}{n}\right)^n = e-\frac{e}{2n}+O\left(\frac{1}{n^2}\right), \qquad n\to\infty,$$

$$\left(1+\frac{1}{n}\right)^{n+1} = e+\frac{e}{2n}+O\left(\frac{1}{n^2}\right), \qquad n\to\infty,$$

$$\left(\frac{\sqrt{n^2+1}+1}{n}\right)^n = e-\frac{e}{6n^2}+O\left(\frac{1}{n^3}\right), \qquad n\to\infty,$$

and

$$\left(\frac{2n+1}{2n-1}\right)^n = e+\frac{e}{12n^2}+O\left(\frac{1}{n^4}\right), \qquad n\to\infty.$$

On the other hand, all four estimates are ridiculously bad, compared to

$$\sum_{k=0}^{n}\frac{1}{k!} = e+O\left(\frac{1}{n!}\right), \qquad n\to\infty.$$

Does the approximation

$$\sum_{k=0}^{n}\frac{1}{k!}$$

have an equally simple geometrical interpretation? The reader is invited to produce one.

References

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A NEW PROPERTY OF THE BERNOULLI NUMBERS

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The Bernoulli numbers B_n are defined as the coefficients in the expansion

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n x^{2n}}{(2n)!}, \qquad (|x| < 2\pi).$$

The B_n's possess many important arithmetical properties and often have proven useful in algebra, topology and analysis. One of the most celebrated of these properties is the

THEOREM (von Staudt-Clausen).

$$B_n = \textit{an integer} + (-1)^n \sum_{(p-1)|2n} \frac{1}{p},$$

where the summation is over all primes p such that $(p-1)|2n$.

In this note we prove a simple result that seems to have gone unnoticed before. If properly interpreted the new property says that the remainders in the von Staudt-Clausen Theorem are bounded on the average.

THEOREM. *We have*

$$\sum_{n \leq x} \left((-1)^n \sum_{(p-1)|2n} \frac{1}{p} \right) = Ax + O(\log\log x), \tag{1}$$

where

$$A = \sum_{p \equiv 1 (\mathrm{mod}\, 4)} \frac{2}{p(p-1)},$$

where the sum is taken over all primes.

Our proof is quite simple and uses only a well-known estimate from elementary prime number theory, namely

$$\sum_{p \leq x} \frac{1}{p} = O(\log\log x),$$

where the sum is taken over primes. A proof of this result, and many others concerning Bernoulli numbers, can be found in Hardy and Wright [**1**].

Proof. If we interchange the order of summation in (1), we obtain

$$\begin{aligned} &\sum_{p \leq 2x+1} \frac{1}{p} \left(\sum_{\substack{n \leq x \\ 2n \equiv O(p-1)}} (-1)^n \right) \\ &= \sum_{p \leq 2x+1} \frac{1}{p} \left(\sum_{t \leq 2x/(p-1)} (-1)^{(p-1)/2 \cdot t} \right) = \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned} \tag{2}$$

where we have partitioned the sum into three parts: Σ_1, Σ_2, Σ_3, containing those terms for which $p \equiv 1 \pmod 4$, and $p \equiv 3 \pmod 4$, $p = 2$ respectively. The summations of Σ_1 and Σ_2 are done by first observing that

$$(-1)^{(p-1)/2\cdot t} = \begin{cases} 1 \text{ if } p \equiv 1 \pmod 4 \\ (-1)^t \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

Hence

$$\Sigma_1 = \sum_{\substack{p \leqq 2x+1 \\ p \equiv 1 \pmod 4}} \frac{1}{p}\left(\sum_{t \leqq (2x/(p-1))} 1\right) = \sum_{\substack{p \leqq 2x+1 \\ p \equiv 1 \pmod 4}} \frac{1}{p}\left(\frac{2x}{p-1} + O(1)\right)$$

$$= x \sum_{\substack{p \leqq 2x+1 \\ p \equiv 1 \pmod 4}} \frac{2}{p(p-1)} + O\left(\sum_{p \leqq 3x} \frac{1}{p}\right)$$

$$= 2x\left(\sum_{p \equiv 1 \pmod 4} \frac{1}{p(p-1)}\right) + O\left(\frac{1}{x}\right) + O(\log\log x)$$

$$= Ax + O(\log\log x).$$

The sum Σ_2 is treated as follows:

$$(4) \qquad \sum_{\substack{p \leqq 2x+1 \\ p \equiv 3 \pmod 4}} \frac{1}{p}\left(\sum_{t \leqq 2x/(p-1)} (-1)^t\right) = O\left(\sum_{\substack{p \leqq 2x+1 \\ p \equiv 3 \pmod 4}} \frac{1}{p}\right) = O(\log\log x).$$

Clearly

$$(5) \qquad \Sigma_3 = \tfrac{1}{2} \sum_{n \leqq x} (-1)^n = O(1).$$

(2), (3), (4) and (5) now give (1). This completes the proof of the theorem.

Remark. A somewhat simpler computation will also give

$$\sum_{n \leqq x}\left(\sum_{(p-1)|2n} \frac{1}{p}\right) = Bx + O(\log\log x),$$

where

$$B = \tfrac{1}{2} + \sum_{p \text{ odd}} \frac{2}{p(p-1)}.$$

Reference

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SYMMETRIES OF THE CAYLEY GROUP TABLE

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It is well known that a finite group is abelian if and only if its Cayley Table is symmetric (about its diagonal running down left to right with the elements in the left column and top row arranged in like order). But when is the table symmetric about its other diagonal and what happens if the arrangement of elements in left column and top row differ? We will answer these questions, but first some definitions.

The elements in the left column and top row of a Cayley Table are called *left multipliers* and *right multipliers* respectively. The table is *symmetric* (*cross-symmetric*) if it is symmetric about its diagonal running down (up) left to right. With reference to its Cayley Table, call a finite group:

(a) *symmetric* (*cross-symmetric*) if there exists *some* ordering of left multipliers and the *same* ordering of right multipliers such that the table is symmetric (cross-symmetric).

(b) *weakly symmetric* (*weakly cross-symmetric*) if there exists *some* ordering of left multipliers and *some* ordering of right multipliers such that the table is symmetric (cross-symmetric).

(c) *strongly symmetric* (*strongly cross-symmetric*) if for *any* ordering of left multipliers and the *same* ordering of right multipliers the table is symmetric (cross-symmetric).

In a, b, c, if the table is simultaneously symmetric and cross-symmetric, the group is *bisymmetric*, *weakly bisymmetric*, *strongly bisymmetric*, respectively.

Remark. A yet stronger symmetry (and cross-symmetry) is possessed by groups of order one or two. The tables of these groups alone are symmetric (and cross-symmetric) no matter how the multipliers are arranged.

THEOREM 1. *A finite group is cross-symmetric if and only if it is abelian of even order.*

Proof. (a) Assume group G is abelian of order $2n$. We let $n = 4$ for convenience. The proof is independent of choice of n. Let x_0 be the identity. Let x_7 be an element of order 2. Pick x_1 and let $x_6 = x_1x_7$, pick x_2 and let $x_5 = x_2x_7$, pick x_3 and let $x_4 = x_3x_7$. Note that $x_i = x_jx_7$ implies $x_j = x_ix_7$. Arrange left and right multipliers in the natural order. The table will be cross-symmetric if $x_ix_k = x_hx_j$ whenever $i + j = k + h = 7$. But, in our example, $x_ix_k = (x_jx_7)(x_hx_7) = (x_jx_h)(x_7x_7) = x_jx_h$. Thus, G is cross-symmetric. (b) Assume G is cross-symmetric. Suppose G is of odd order. Then the identity e is the only idempotent and must appear in the center of the table and the center of the left and right multipliers. Let a, e, b in order be the three central elements in the left column and top row. The center of the table is

	a	
a	e	b
	b	

Hence, $a = b$, a contradiction. Thus G is of even order and has an element of order 2. We show G is abelian. List the elements in order: $x_0, x_1, \cdots, x_{2n-1}$. If x_r is the identity, then x_s has order 2, where $r + s = 2n - 1$. If $k + h = 2n - 1$, then $x_s x_k = x_h x_r = x_h = x_r x_h = x_k x_s$. Thus x_s commutes with all elements and $k + h = 2n - 1$ implies $x_s x_k = x_h$. Consider $x_i x_k$, for arbitrary i and k. Let $j = 2n - 1 - i$ and $h = 2n - 1 - k$. Then $x_i x_k = (x_s x_j)(x_s x_h) = x_j x_h = x_k x_i$, and G is abelian.

COROLLARY. *A finite group is bisymmetric if and only if it is abelian of even order.*

THEOREM 2. *A group is weakly symmetric if and only if it is weakly cross-symmetric.*

Proof. A Cayley Table with right multipliers: $x_0, x_1, \cdots, x_{n-1}, x_n$ is symmetric if and only if it is cross-symmetric with right multipliers: $x_n, x_{n-1}, \cdots, x_1, x_0$.

THEOREM 3. *A finite group is weakly symmetric if and only if it is abelian.*

Proof. If G is abelian, it is symmetric and hence weakly symmetric. If G is weakly symmetric, we show $xy = yx$ for x, y in G. From matrix theory we know a symmetric matrix remains symmetric when two rows and the same two columns are permuted. By such permutations, we let the multiplying row begin with e, x, y. For some a, b, c, the upper left corner of the table is

	e	x	y	
a	a	ax	ay	
b	b	bx	by	
c	c	cx	cy	

By symmetry, $by = cx$, $c = ay$, $b = ax$. Thus $(ax)y = by = cx = (ay)x$, and $xy = yx$. Thus G is abelian.

COROLLARY. *A finite group is weakly cross-symmetric if and only if it is abelian.*

Still, a group may satisfy Theorem 3 and its corollary and not be weakly bisymmetric.

THEOREM 4. *A weakly bisymmetric group is of even order.*

Proof. Suppose G is bisymmetric of odd order. Consider the three central left and right multipliers and their products.

	x	y	z
a		ay	
b	bx		bz
c		cy	

Then $bx = ay = bz$ and the same element appears twice in the same row. Therefore, G is of even order.

By Theorems 3 and 4 and the corollary to Theorem 1, we obtain the following corollary:

COROLLARY. *A finite group is weakly bisymmetric if and only if it is abelian of even order.*

A group is strongly symmetric if and only if it is abelian, but what groups are strongly cross-symmetric or bisymmetric? We show that both classes consist of but three groups.

LEMMA. *If group G is strongly cross-symmetric, then the order of G is less than* 5.

Proof. Let G be strongly cross-symmetric and of order greater than four. Consider the upper right corner of its table.

	a	b	c
	x	y	
			y
		z	x

A permutation of columns b and c and the last two rows preserves cross-symmetry. Since the order of G is greater than four, the row permutation does not affect row three. We now have

	a	c	b
	x		y
		y	
		x	z

Since $z \neq x$, we have a contradiction.

THEOREM 5. *The only strongly cross-symmetric groups are the four group and groups of order one or two.*

Proof. By examination the groups mentioned above are strongly cross-symmetric, while cyclic groups of order three and four are not. For example, consider the integers modulo three under addition:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The lemma completes the proof.

COROLLARY. *The only strongly bisymmetric groups are the four group and groups of order one or two.*

Let us summarize our results with a table.

	Symmetric	Cross-symmetric	Bisymmetric
Weakly	Abelian	Abelian	Abelian of even order
	Abelian	Abelian of even order	Abelian of even order
Strongly	Abelian	Order 1 or 2 Four group	Order 1 or 2 Four group

HAIR

KATHARINE O'BRIEN, Portland, Maine

DESCARTES had bangs,
a shoulder-length hair-do —
looked like a pirate
or swash-buckling dare-do.

FERMAT's hair longish
and parted in the middle —
a quizzical expression
(Fermat's Last Riddle?)

LEIBNIZ wore a wig
in a formal sort of way
like a judge in a courtroom
or an actor in a play.

LAGRANGE — patrician nose,
a medal on a fob —
his white hair cut
in a page-boy bob.

FOURIER had ringlets,
nature's own style —
on his cherubic face
a captivating smile.

GAUSS favored side-burns,
a hat with no brim —
finely chiseled features
distinguishing him.

LOBACHEVSKY — clean-shaven
with trimmed black hair —
medals on his chest
and aloofness in his air.

RIEMANN — granny glasses
and a thick black beard —
learned and sincere
was how Riemann appeared.

CANTOR — a goatee
and mustache foliation —
a most impressive figure
in any aggregation.

EINSTEIN looked amused
at some abstruse reflection —
his hair shooting off
in every direction.

COROLLARY. *The only strongly bisymmetric groups are the four group and groups of order one or two.*

Let us summarize our results with a table.

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ON THE REPRESENTATION OF INTEGERS

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The purpose of this paper is to give a generalization of the following

THEOREM (H. F. Scherk). *If* $\{p_i\}$ *is the sequence of all primes with* $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $p_4 = 5, \cdots$, *then every prime p can be written, with signs suitably chosen, either in the form*

$$p = p_m \pm p_{m-1} \pm \cdots \pm p_1$$

or in the form

$$p = 2p_m \pm p_{m-1} \pm \cdots \pm p_1$$

where $p = p_{m+1}$.

The above theorem suggests the following

DEFINITION. *A strictly increasing infinite sequence* $\{s_i\}$ *of positive integers is a Scherk sequence if and only if every positive integer n (not just primes) can be written, with signs suitably chosen, either in form* (A) *or in form* (B),

(A) $n = s_m \pm s_{m-1} \pm \cdots \pm s_1$,

(B) $n = 2s_m \pm s_{m-1} \pm \cdots \pm s_1$,

where $s_{m+1} \geqq n > s_m$.

LEMMA. *Let* $\{s_i\}$ *be a strictly increasing infinite sequence of positive integers with* $s_1 = 1$ *and satisfying* $s_{m+1} \leqq 1 + \sum_1^m s_i$, *for* $m = 1, 2, 3, \cdots$. *If signs are suitably chosen, then the expression* $s_m \pm s_{m-1} \pm \cdots \pm s_1$ *assumes every even (odd) integer from* $\sum_1^m s_i$ *to* 0 (1) *if* $\sum_1^m s_i$ *is even (odd), for* $m = 1, 2, 3, \cdots$.

REMARK. The expression $s_m \pm s_{m-1} \pm \cdots \pm s_1$ has the same parity as $\sum_1^m s_i$, i.e., either even or odd, independently of how the plus and minus signs are affixed.

Proof. The proof goes by induction on the subscript *m*. It is obvious that $s_2 = 2$ and the lemma holds for $m = 2$, since $2 + 1 = 3, 2 - 1 = 1$. Assume that it holds for $m = k$, then we must show that the expression $s_{k+1} \pm s_k \pm \cdots \pm s_1$ assumes every even (odd) integer from $\sum_1^{k+1} s_i$ to 0 (1) if $\sum_1^{k+1} s_i$ is even (odd). We distinguish the following:

Case 1. Suppose $\sum_1^k s_i$ is even. Since the lemma holds for $m = k$ and $\sum_1^k s_i$ is even, hence the expression $s_k \pm s_{k-1} \cdots \pm s_1$ assumes every even integer from $\sum_1^k s_i$ to 0. It follows that the expression $s_{k+1} \pm s_k \pm s_{k-1} \pm \cdots \pm s_1$ assumes every even (odd) integer from $\sum_1^{k+1} s_i$ to $(s_{k+1} - \sum_1^k s_i)$, if $\sum_1^{k+1} s_i$ is even (odd). By assumption, $(s_{k+1} - \sum_1^k s_i)$ is an even integer less than 1, if s_{k+1} is even. If s_{k+1} is odd, then $(s_{k+1} - \sum_1^k s_i) \leqq 1$.

Case 2. Suppose $\sum_1^k s_i$ is odd. The same argument is used. Thus the proof of the lemma is complete.

Now we are ready to prove the

MAIN THEOREM. *Let $\{s_i\}$ be a strictly increasing infinite sequence of positive integers with $s_1 = 1$. Then $\{s_i\}$ is a Scherk sequence if and only if it satisfies*

(1) *whenever s_t is even, then $\sum_1^t s_i$ is odd and $s_{t+1} - s_t = 1$, and*

(2) $s_{m+1} \leqq 1 + \sum_1^m s_i$, *for* $m = 1, 2, 3, \cdots$

Proof. First we show sufficiency. Assume that $s_{m+1} \geqq n > s_m$. We separate into:

Case 1. Suppose n and $\sum_1^m s_i$ have the same parity, i.e., both even or both odd. We have $\sum_1^m s_i \geqq n$. From the lemma, the expression $s_m \pm s_{m-1} \pm \cdots \pm s_1$ assumes every even (odd) integer from $\sum_1^m s_i$ to 0 (1), if $\sum_1^m s_i$ is even (odd). Hence if we choose signs properly, then $n = s_m \pm s_{m-1} \pm \cdots \pm s_1$.

Case 2. Suppose n is odd and $\sum_1^m s_i$ is even. It follows from the assumption that s_m is not even. Again, from the lemma, the expression $s_m + (s_m \pm s_{m-1} \pm \cdots \pm s_1)$ assumes every odd integer from $(s_m + \sum_1^m s_i)$ to s_m. Hence $n = 2s_m \pm s_{m-1} \pm \cdots \pm s_1$, for some suitably chosen signs.

Case 3. Suppose n is even and $\sum_1^m s_i$ is odd. If s_m is even, then by assumption $s_{m+1} = s_m + 1$. This implies that there is no even integer n such that $s_{m+1} \geqq n > s_m$. So s_m is odd. The lemma shows that the expression $s_m + (s_m \pm s_{m-1} \pm \cdots \pm s_1)$ assumes every even integer from $(s_m + \sum_1^m s_i)$ to $s_m + 1$. Hence $n = 2s_m \pm s_{m-1} \pm \cdots \pm s_1$, for some suitably chosen signs.

Second we show necessity.

(A) Suppose that (1) fails to hold, i.e., there exists an integer j such that s_j is even, but $\sum_1^j s_i$ is even or $(s_{j+1} - s_j) > 1$. We consider:

Case 1. Suppose $\sum_1^j s_i$ is even. It is easy to show that if n is any odd integer such that $s_{j+1} \geqq n > s_j$, then n cannot be expressed either in form A or in form B, regardless of how the signs are selected.

Case 2. Suppose that $(s_{j+1} - s_j) > 1$. From above, $\sum_1^j s_i$ must be odd. It is easy to see that $(s_j + 2)$ cannot be expressed in the desired form. Hence if $\{s_i\}$ is a Scherk sequence, then whenever s_t is even, $\sum_1^t s_i$ is odd and $s_{t+1} - s_t = 1$.

(B) Suppose that (2) does not hold, i.e., there exists $k \geqq 1$ such that $s_{k+1} > 1 + \sum_1^k s_i$. Choose an integer n with $s_{k+1} \geqq n > \sum_1^k s_i$ whose parity is different from that of $(s_k + \sum_1^k s_i)$. Then it is easily seen that n cannot be represented in the desired form. Thus the proof of the main theorem is complete.

COROLLARY. *If $\{p_i\}$ is the sequence of all primes with $p_1 = 1$, then $\{p_i\}$ is a Scherk sequence.*

REMARK. Scherk's theorem is included in the above corollary.

Proof. By Bertrand's postulate, $p_{k+1} \leqq 2p_k$ for $k \geqq 1$ with strict inequality for $k > 1$. Making use of this fact, it is easy to establish by induction that p_{k+1}

$\leqq 1 + \sum_1^k p_i$, which is (2) in our main theorem. Since p_k is always odd when $k > 2$, (1) is automatically fulfilled. Thus we have proved that the sequence of all primes begining with 1 is a Scherk sequence.

COROLLARY. *The Fibonacci sequence* $\{f_i\}$ *with* $f_1 = 1, f_2 = 2, \cdots$, *is not a Scherk sequence.*

Proof. We have $f_5 = 8$ and $f_6 = 13$, so by the main theorem the Fibonacci sequence is not a Scherk sequence.

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A COUNTEREXAMPLE IN MATRIX ANALYSIS

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If A and B are real or complex square matrices of the same order, then

$$(*) \qquad \lim_{m\to\infty} (AB)^m = (\lim_{j\to\infty} A^j)(\lim_{k\to\infty} B^k)$$

is true, provided the two limits on the right exist *and* $AB = BA$. We construct a simple [counter] example showing that the result (*) *may* fail without commutativity; an interesting aspect of the example is the fact that it is "discovered" by considering a particular interpretation of the matrices A and B.

Consider a (memoryless) three-state system which makes transitions in response to one of two input signals (say α and β) and whose transition probabilities (degenerate in this case) are as indicated in the diagrams:

The corresponding matrices of transition probabilities are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\leqq 1 + \sum_1^k p_i$, which is (2) in our main theorem. Since p_k is always odd when $k > 2$, (1) is automatically fulfilled. Thus we have proved that the sequence of all primes begining with 1 is a Scherk sequence.

COROLLARY. *The Fibonacci sequence* $\{f_i\}$ *with* $f_1 = 1, f_2 = 2, \cdots$, *is not a Scherk sequence.*

Proof. We have $f_5 = 8$ and $f_6 = 13$, so by the main theorem the Fibonacci sequence is not a Scherk sequence.

Acknowledgment. The author wishes to thank the referee for his valuable suggestions.

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A COUNTEREXAMPLE IN MATRIX ANALYSIS

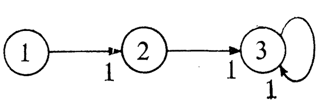
B. B. WINTER, University of Ottawa, Canada

If A and B are real or complex square matrices of the same order, then

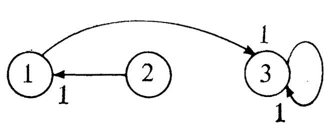
$$(*) \qquad \lim_{m\to\infty} (AB)^m = (\lim_{j\to\infty} A^j)(\lim_{k\to\infty} B^k)$$

is true, provided the two limits on the right exist *and* $AB = BA$. We construct a simple [counter] example showing that the result (*) *may* fail without commutativity; an interesting aspect of the example is the fact that it is "discovered" by considering a particular interpretation of the matrices A and B.

Consider a (memoryless) three-state system which makes transitions in response to one of two input signals (say α and β) and whose transition probabilities (degenerate in this case) are as indicated in the diagrams:



α



β

The corresponding matrices of transition probabilities are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A^n is the n-step transition matrix, for the case where signal α is given n times. Similarly, it is easy to see that AB is the two-step transition matrix for the case where signal α is given first, and is followed by signal β. Let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Still viewing all matrices as transition matrices, it is quite obvious (without any computation) that T is idempotent and $A^2 = A^3 = \cdots = T$ and $B^2 = B^3 = \cdots = T$ and, furthermore, that

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and that this matrix is also idempotent. Hence

$$\lim_{m\to\infty} (AB)^m = AB \neq T = (\lim_{j\to\infty} A^j)(\lim_{k\to\infty} B^k).$$

In the above example, $AB \neq BA$ and equation (*) is not true. But, as pointed out by the referee, commutativity is *not really necessary* for (*) to hold. If B is as above but

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

then commutativity fails (because now α followed by β forces the system into state 3, whereas β followed by α forces it into state 1), but nonetheless equation (*) is true: $(\lim A^j)\ (\lim B^k) = \lim(AB)^m = AB$ and $(\lim B^k)\ (\lim A^j) = \lim(BA)^m = BA$. Another noncommuting example (based on a comment by the referee) is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Again (*) is true and, in addition, $(AB)^m$ and $(BA)^m$ have the same limit; in fact, for $m \geqq 3$, A^m, B^m, $(A^m)(B^m)$, $(AB)^m$, and $(BA)^m$ are all matrices with 1's in the fourth column and 0's elsewhere.

CONGRUENCE PROPERTIES OF CERTAIN RESTRICTED PARTITIONS

K. THANIGASALAM, Pennsylvania State University

It was proved by Ramanujan (see [**1**] and [**2**]) that the number $p(n)$ of unrestricted partitions of a natural number n satisfies the following congruences:

(1) $$p(n) \equiv 0 \pmod 5 \quad \text{if } n \equiv 4 \pmod 5$$

(2) $$p(n) \equiv 0 \pmod 7 \quad \text{if } n \equiv 5 \pmod 7$$

(3) $$p(n) \equiv 0 \pmod{11} \quad \text{if } n \equiv 6 \pmod{11}.$$

It is the object of this paper to show that the same congruences are satisfied by certain restricted partitions. Let $p_k^*(n)$ denote the number of partitions of n into parts not divisible by k where $k \geqq 2$ is a given natural number. Then we prove a general theorem, from which by the aid of (1), (2), and (3) it will follow that

(4) $$p_5^*(n) \equiv 0 \pmod 5 \quad \text{if } n \equiv 4 \pmod 5$$

(5) $$p_7^*(n) \equiv 0 \pmod 7 \quad \text{if } n \equiv 5 \pmod 7$$

(6) $$p_{11}^*(n) \equiv 0 \pmod{11} \quad \text{if } n \equiv 6 \pmod{11}.$$

We shall now state the main theorem.

THEOREM. *Let* $k \geqq 2$, *and* l *be an integer satisfying* $0 \leqq l \leqq k-1$. *If* $p(km+l) \equiv 0 \pmod k$ *for* $m = 0,1,2,\cdots$, *then* $p_k^*(km+l) \equiv 0 \pmod k$ $(m = 0,1,2,\cdots)$.

Conversely, if $p_k^*(km+l) \equiv 0 \pmod k$ $(m = 0,1,2,\cdots)$, then $p(km+l) \equiv 0 \pmod k$ $(m = 0,1,2,\cdots)$.

Proof of Theorem. First suppose that $p(km+l) \equiv 0 \pmod k$ $(m = 0,1,2,\cdots)$. The generating function

$$1 + \sum_{n=1}^{\infty} p_k^*(n)x^n \text{ of } p_k^*(n) \text{ is } \prod_{m=1}^{\infty}(1-x^{km}) / \prod_{j=1}^{\infty}(1-x^j).$$

Thus

(7) $$\left\{1 + \sum_{r=1}^{\infty} p(r)x^r\right\}\left\{\prod_{m=1}^{\infty}(1-x^{km})\right\} = 1 + \sum_{n=1}^{\infty} p_k^*(n)x^n.$$

By Euler's product formula

$$\prod_{m=1}^{\infty}(1-z^m) = 1 + \sum_{m=1}^{\infty}(-1)^m\{z^{m(3m+1)/2} + z^{m(3m-1)/2}\}.$$

Taking $z = x^k$ in this, we have

(8) $$\prod_{m=1}^{\infty}(1-x^{km}) = 1 + \sum_{m=1}^{\infty}(-1)^m\{x^{km(3m+1)/2} + x^{km(3m-1)/2}\},$$

where the integer exponents $km(3m+1)/2$, $km(3m-1)/2$ of x are both divisible by k for $m = 1, 2, 3, \cdots$. Hence, if $n \equiv l \pmod{k}$, the coefficient of x^n on the left hand side of (7) will be of the form $p(r_1) \pm p(r_2) \pm \cdots \pm p(r_t)$, where each r_i satisfies $r_i \equiv l \pmod{k}$. Since this sum is equal to $p_k^*(n)$, and $p(r_i) \equiv 0 \pmod{k}$ for each r_i, we see that $p_k^*(n) \equiv 0 \pmod{k}$, as required. Now suppose that $p_k^*(km+l) \equiv 0 \pmod{k}$ $(m = 0, 1, 2, \cdots)$. We have

$$1 + \sum_{n=1}^{\infty} p(n)x^n = \frac{1}{\prod_{j=1}^{\infty} (1 - x^j)}$$

$$= \left\{1 + \sum_{r=1}^{\infty} p_k^*(r)x^r\right\}\left\{\prod_{m=1}^{\infty} (1 - x^{km})^{-1}\right\}$$

$$= \left\{1 + \sum_{r=1}^{\infty} p_k^*(r)x^r\right\}\left\{\prod_{m=1}^{\infty} (1 + x^{km} + x^{2km} + \cdots)\right\}.$$

If $n \equiv l \pmod{k}$, the coefficient of x^n on the right hand side will be of the form $p_k^*(r_1) + p_k^*(r_2) + \cdots p_k^*(r_s)$, where $r_i \equiv l \pmod{k}$ for $i = 1, 2, \cdots s$. Since $p_k^*(r_i) \equiv 0 \pmod{k}$ for each r_i, it follows that $p(n) \equiv 0 \pmod{k}$. This completes the proof of the theorem.

REMARKS. $p(n)$ is known to satisfy the following congruences:

(9) $$p(n) \equiv 0 \pmod{5^a} \quad \text{if } 24n \equiv 1 \pmod{5^a}$$

(10) $$p(n) \equiv 0 \pmod{7^2} \quad \text{if } 24n \equiv 1 \pmod{7^2}$$

(11) $$p(n) \equiv 0 \pmod{11^c} \quad \text{if } 24n \equiv 1 \pmod{11^c}.$$

Here (9) is due to G. N. Watson [**3**], (10) is due to L. J. Mordell [**4**], and (11) is due to A. O. L. Atkin [**5**].

COROLLARY. *In addition to* (4), (5) *and* (6), $p_k^*(n)$ *satisfies the following congruences*:

(12) $$p_k^*(n) \equiv 0 \pmod{5^a} \quad \text{if } 24n \equiv 1 \pmod{5^a}$$

(13) $$p_k^*(n) \equiv 0 \pmod{7^2} \quad \text{if } 24n \equiv 1 \pmod{7^2}$$

(14) $$p_k^*(n) \equiv 0 \pmod{11^c} \quad \text{if } 24n \equiv 1 \pmod{11^c}.$$

Since the congruence $ax \equiv b \pmod{k}$ with $(a, k) = 1$ has a unique solution modulo k, we see that in view of the main theorem of this paper, the results (4), (5), (6), (12), (13) and (14) follow from (1), (2), (3), (9), (10) and (11), respectively.

Acknowledgement. I am indebted to the referee for pointing out an error, and making useful suggestions.

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1. S. Ramanujan, Some properties of $p(n)$, the number of partitions of n, Proc. Cambridge Philos. Soc., 19 (1919) 207–210.

2. ———, Congruence properties of partitions, Math Z., 9 (1921) 147–153.

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4. L. J. Mordell, Note on certain modular relations considered by Ramanujan, Darling and Rogers, Proc. London Math. Soc., 20 (1922) 408–416.

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A SUFFICIENT CONDITION FOR n-SHORT-CONNECTEDNESS

BRUCE HEDMAN, University of Washington

In this article we will consider only graphs without loops or multiple edges. If a and b are vertices of a graph, then (a, b) denotes the edge between them. A graph is k-valent if the valence of every vertex of the graph is $\geqq k$. A graph G is n-connected if for every two vertices a, b of G there are at least n paths in G with endpoints a, b which are otherwise disjoint. The length of a path is the number of edges composing the path. A graph G is n-short-connected if for every two vertices a, b of G there are at least n paths in G with endpoints a, b each of length $\leqq 2$, i.e., for every two vertices a, b of G either there are n paths of length 2 with endpoints a, b, or there are $(n-1)$ paths of length 2 with endpoints a, b and $(a, b) \in G$.

Short-connectedness is a stricter condition than "regular" connectedness since it not only demands the existence of n disjoint paths between every two vertices of the graph but also restricts the length of these paths rather severely. However, as we will see, the sufficient condition upon the valency of a graph for n-short-connectedness is only slightly more demanding than the well-known (see [**1**]) condition for n-"regular" connectedness. The threefold purpose of this article is to introduce short-connectedness and a consequence of it, to establish a sufficient condition for n-short-connectedness, and to illustrate a new proof of and a remark concerning the Chartrand-Harary theorem [**1**].

Chartrand and Harary observed the following (my rewording):

THEOREM 1: *For integers $p > n > 1$, define $k_1 =](p+n-2)/2[$. If graph G contains p vertices and is k_1-valent, then G is n-connected.*

$$\text{For } x > 0, \quad]x[= \begin{cases} x, & \text{if } x \text{ is an integer,} \\ x', & \text{where } x' \text{ is the integer satisfying } 0 < x' - x < 1, \text{ if } x \text{ is not an integer.} \end{cases}$$

Theorem 2 is the main part of this article.

THEOREM 2. *For integers $p > n > 1$, define $k_0 =](p+n-1)/2[$. If graph G contains p vertices and is k_0-valent, then G is n-short-connected.*

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Theorem 2 is the main part of this article.

THEOREM 2. *For integers $p > n > 1$, define $k_0 =](p+n-1)/2[$. If graph G contains p vertices and is k_0-valent, then G is n-short-connected.*

Observe that only when $p + n$ is even does Theorem 2 demand more than Theorem 1. A proof of Theorem 1 follows from that of Theorem 2, and will be indicated thereafter.

Proof of Theorem 2. Choose any two vertices a, b of G. Define T_a to be the set containing a and all vertices of G which are joined to a by an edge, except b if $(a, b) \in G$. Similarly define $T_b = \{b\} \cup \{v(\neq a) \in G : (v, b) \in G\}$. Let $j = \text{card}(T_a \cap T_b)$.

CASE 1. $(a, b) \notin G$. Since $\text{val}(a) \geqq k_0$ and $\text{val}(b) \geqq k_0$, then $p \geqq \text{card}(T_a \cup T_b) \geqq 2(k_0 + 1) - j$. Since $k_0 =](p + n - 1)/2[$ and $]x[\geqq x$, $p \geqq (p + n - 1) + 2 - j$, thus $n + 1 \leqq j$. Since $v \in T_a \cap T_b$ means $(v, a), (v, b) \in G$, G is $(n + 1)$-short-connected.

CASE 2. $(a, b) \in G$. In this case, $\text{card}(T_a) \geqq k_0$, $\text{card}(T_b) \geqq k_0$, and $p \geqq \text{card}(T_a \cup T_b) \geqq 2k_0 - j$. Thus $p \geqq p + n - 1 - j$ and $j \geqq n - 1$. Since $(a, b) \in G$, G is n-short-connected.

To prove Theorem 1, observe that had k_1 been substituted for k_0, Case 1 would have shown that G was n-connected and Case 2 would have shown that G was at least $(n - 1)$-connected. The following continues Case 2 to find the nth disjoint path and arrives at an additional observation regarding Theorem 1.

If $\text{card}(T_a) > k_1$, then $p \geqq \text{card}(T_a \cup T_b) > 2k_1 - j$ and $n - 2 < j$ (strict inequality); thus G is n-connected. If card $(T_a) = k_1$, then let $x_0 \in T_a - T_b - a$. (If this is empty, then $T_a - a \subset T_b$ and $j = k_1 - 1 \geqq](p + n - 2)/2[- 1 \geqq](2n-1)/2[-1 = n - 1$. Thus $(a, b) \in G$ implies G *is* n-connected.) Since $\text{val}(x_0) \geqq k_1$ and $\text{card}(T_a) = k_1$, there exists $v_0 \in G - T_a$ such that $(x_0, v_0) \in G$. Since $x_0 \notin T_b$, $v_0 \neq b$. If $v_0 \in T_b - b$, then $(a, x_0), (x_0, v_0), (v_0, b)$ constitute another path between a and b, and G is n-connected. If $v_0 \notin T_b - b$, then, since $v_0 \notin T_a$, $\text{card}(T_a \cup T_b) < p$ (strict inequality), and then, as before, $p > 2k_1 - j$ and G is n-connected.

Observe that in the proof of Theorem 1 we showed either $j \geqq n - 1$ or that $j \geqq n - 2$ and there exists a path of length 3 between a and b. Thus the conditions of Theorem 1 insure that between any two vertices of G there are n disjoint paths of length $\leqq 3$.

We conclude this article with an implication of n-short-connectedness. It may be proved by a routine inductive argument once one notes that an $(n + 1)$-short-connected graph becomes n-short-connected when one vertex and all edges containing it are deleted.

THEOREM 3. *If a graph G is n-short-connected, then between any two vertices of G there exists a path of length exactly n.*

My thanks to Dr. Branko Grünbaum of the University of Washington who suggested a valuable simplification in my proof of Theorem 2.

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A GENERALIZATION OF EISENSTEIN'S CRITERION

HOWARD CHAO, Purdue University

Let $q(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with integer coefficients a_i. Then Eisenstein's Criterion is:

If p is a prime such that $p \nmid a_n$, $p \mid a_i$ for $i < n$, and $p^2 \nmid a_0$, then $q(x)$ is irreducible over $Q[x]$. (See [**1**], p. 121.)

We prove the following more general statement:

If p is a prime such that for some l and k with $l \neq k, p \nmid a_l$, $p^2 \nmid a_k$, $p \mid a_i$ for $i \neq l$ and if $q(x)$ is reducible into the product of two polynomials in $Q[x]$, then the degree of one of these factors must be $\geqq |l - k|$.

It is clear that Eisenstein's Criterion follows as a special case when $l = n$ and $k = 0$. There are some other corollaries that will be stated later.

Proof. Let $q(x) = r(x)s(x)$ be a factorization of $q(x)$ where

$$r(x) = \sum_{i=0}^{m} b_i x^i, \qquad s(x) = \sum_{i=0}^{n-m} c_i x^i.$$

We need only consider factors with integer coefficients b_i and c_i, since any integer polynomial factorable into two rational polynomials is factorable into two integer polynomials which are associates of the rational factors, by Gauss' Lemma ([**1**], p. 120).

Now let $[q]_p(x)$ be the polynomial $q(x)$ with coefficients reduced mod p and let $[a_i]_p$ denote the residue of a_i mod p. Then

$$[q]_p(x) = \sum_{i=0}^{n} [a_i]_p x^i = [a_l]_p x^l = [r]_p(x)\,[s]_p(x).$$

By unique factorization in $Z[x]$, $[r]_p(x)$ and $[s]_p(x)$ must be of the form $[b_u]_p x^u$ and $[c_t]_p x^t$, respectively, where $u + t = l$. Thus it follows that $p \mid b_i$ for $i \neq u$ and $p \mid c_i$ for $i \neq t$.

We break the argument up into two cases.

Case I: $k > l$. Since $l = u + t$, then $k > u$ and $k > t$. Either $m + t \geqq k$ or $n - m + u \geqq k$; because otherwise the coefficients of c_t and b_u in $a_k = \sum_{i=0}^{k} b_i c_{k-i}$ are zero, and hence a_k would be divisible by p^2. If $n - m + u \geqq k$, by renaming the factor polynomials and coefficients we can get $m + t \geqq k$. So without loss of generality:

$$m + t \geqq k, \; m \geqq k - t, \text{ where } 0 \leqq t \leqq l.$$

Therefore $m \geqq k - l$.

Case II: $l > k$. First, $k \geqq \min(u, t)$; for if $u > k$ and $t > k$ then $a_k = \sum_{i=0}^{k} b_i c_{k-i}$ is again divisible by p^2, since $p \mid b_i$, $p \mid c_j$ for $i < u$, $j < t$. Without loss of generality we may assume $k \geqq t$.

But since $m \geqq u = l - t$ and $k \geqq t$ then $m \geqq l - k$. So in general $m \geqq |l-k|$.

It should be noted that the theorem gives no information when $|l-k| \leqq n/2$ for even n and when $|l-k| \leqq (n+1)/2$ for odd n.

An interesting corollary is that Eisenstein's Criterion can be restated with $p \nmid a_0$ and $p^2 \nmid a_n$, and the same conclusion holds. (This also can be easily shown using the normal form of Eisenstein's Criterion.)

This generalization leads to a prediction for the number of factors of a given degree.

COROLLARY. *Given a polynomial satisfying the conditions specified in the theorem and* $|l-k| > n/2$ *if* n *is even,* $|l-k| > (n+1)/2$ *if* n *is odd. Then there are at most* $n - |l-k|$ *rational factors of degree* 1.

Proof. Assume there are more than $n - |l-k|$ 1st degree factors; then take the product of $n - |l-k| + 1$ of them as one factor and the product of all other factors as the second factor. Then neither factor has degree $\geqq |l-k|$, contradicting the above theorem.

Similar statements can be made about higher degree factors. Clearly there can be no factors of degree y if $n - |l-k| < y < |l-k|$.

It is interesting to note that all factors are similar in form to the original polynomial in that p divides all coefficients but one, and in addition, one of the factors satisfies completely the hypothesis of the theorem.

Reference

1. I. N. Herstein, Topics in Algebra, Blaisdell, New York, 1964.

AN EXTENSION OF BROCARD GEOMETRY

PAUL SIDENBLAD, University of Santa Clara

It is my aim to present three theorems concerning points related to the Brocard points of a triangle, whose vertices will be labeled A, B, and C.

We begin by constructing three circles: the circle tangent to AB at A and passing through C, the circle tangent to BC at B and passing through A, and the circle tangent to AC at C and passing through B (Figure 1). The centers of these circles will be labeled E, F and G, respectively. It can be shown that these circles have a point M in common, called the direct Brocard point of triangle ABC, and that $\angle MAB$

Case II: $l > k$. First, $k \geqq \min(u, t)$; for if $u > k$ and $t > k$ then $a_k = \sum_{i=0}^{k} b_i c_{k-i}$ is again divisible by p^2, since $p \mid b_i$, $p \mid c_j$ for $i < u$, $j < t$. Without loss of generality we may assume $k \geqq t$.

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We begin by constructing three circles: the circle tangent to AB at A and passing through C, the circle tangent to BC at B and passing through A, and the circle tangent to AC at C and passing through B (Figure 1). The centers of these circles will be labeled E, F and G, respectively. It can be shown that these circles have a point M in common, called the direct Brocard point of triangle ABC, and that $\angle MAB$

$= \angle MBC = \angle MCA$. It can also be shown that the circle tangent to AC at A and passing through B, the circle tangent to AB at B and passing through C, and the circle tangent to BC at C and passing through A, also have a point, N, in common, called the indirect Brocard point of triangle ABC. For the point N, we have that $\angle NAC = \angle NBA = \angle NCB = \angle MAB = \angle MBC = \angle MCA$ [1].

If we construct the triangle whose vertices are E, F, and G, we arrive at the following:

THEOREM 1. $\triangle ABC \approx \triangle EFG$.

Since the circle centered at E is tangent to AB at A, $\widehat{AC}$ of that circle equals $2\angle CAB$. Since EF is the line of centers of two intersecting circles, it bisects $\widehat{MA}_E$ ($\widehat{MA}$ of the circle centered at E). Therefore $\angle MEF = \frac{1}{2}\widehat{MA}_E$. Similarly, $\angle MEG = \frac{1}{2}\widehat{MC}_E$. Thus $\angle FEG = \frac{1}{2}\widehat{AC}_E$. However, since the circle centered at E is tangent to AB at A, and passes through C by construction, we have that $\angle CAB = \frac{1}{2}\widehat{AC}_E$. Therefore $\angle GEF = \angle CAB$. Similarly, we can show that $\angle EFG = \angle ABC$, and $\angle FGE = \angle BCA$. Therefore $\triangle ABC \approx \triangle EFG$.

THEOREM 2. *The direct Brocard point of* $\triangle ABC$ *is also the direct Brocard point of* $\triangle EFG$.

Clearly,

$$\angle MAB = \tfrac{1}{2}\widehat{MA}_E = \angle MEF$$
$$\angle MBC = \tfrac{1}{2}\widehat{MB}_F = \angle MFG$$
$$\angle MCA = \tfrac{1}{2}\widehat{MC}_G = \angle MGE.$$

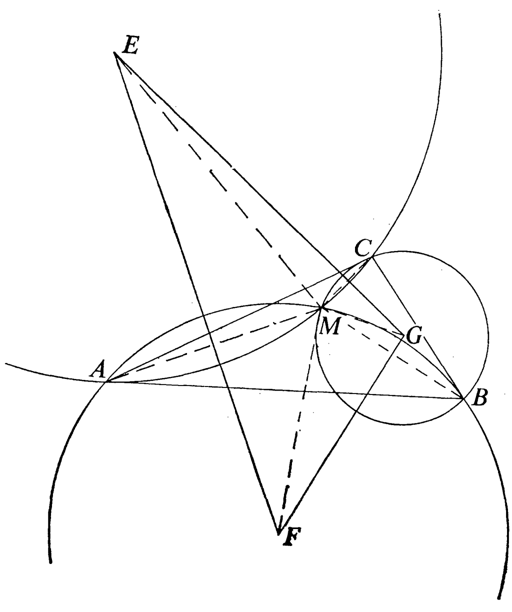

FIG. 1.

Since $\angle MAB = \angle MBC = \angle MCA$, we have $\angle MEF = \angle MFG = \angle MGE$, and the proof is complete.

THEOREM 3. *The circumcenter of* $\triangle ABC$ *is the indirect Brocard point of* $\triangle EFG$. (See Figure 2, where the circumcenter of ABC has been labeled O.)

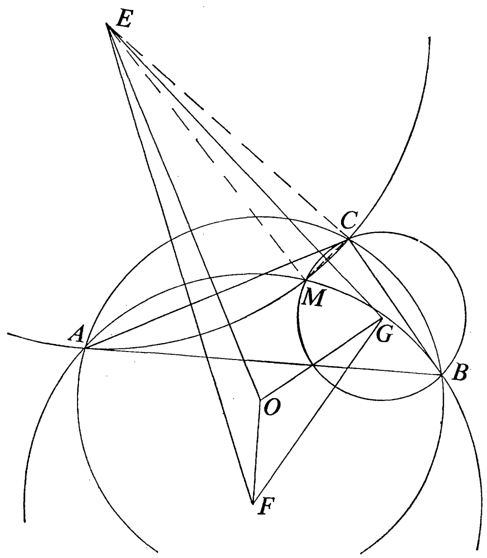

FIG. 2.

Noticing that EO is the line of centers of two circles which have AC as a common chord, we have that EO is the perpendicular bisector of AC and $\angle CEO = \angle GEF$, and hence $\angle CEG = \angle OEF$. Noticing now that EG is the line of centers of two circles having CM as a common chord, we see that EG is the perpendicular bisector of CM, so that we have $\angle CEG = \angle MEG = \angle OEF$. Hence $\angle MEF = \angle GEO$, and similarly we have that $\angle MFG = \angle EFO$, and $\angle MGE = \angle FGO$. This and the isogonality of the Brocard points assure us that the proof is complete.

As a final note, it is worth mentioning that three entirely analogous theorems can be stated and proved by beginning with the indirect rather than the direct Brocard point. Simply exchange B and C, relabel the centers of the three Brocard circles, change the label of the point M to "N," and exchange the words, "direct Brocard point" with "indirect Brocard point."

Partially supported by NSF Grant GY 9923 Undergraduate Research Participation Program in Mathematics, summer 1972, University of Santa Clara. The research was guided by Professor Verner E. Hoggatt, Jr., California State University, San Jose.

Reference

1. Nathan Altshiller-Court, College Geometry, Barnes & Noble, New York, 1952.

THE INFINITUDE OF PRIMES

CHARLES W. TRIGG, San Diego, California

In establishing the infinitude of primes, Euclid (*Elements*, Book IX, Proposition 20) employed the product of all primes $\leqq p$ in the expression

$$Q = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \cdots \cdot p + 1 .$$

From this he concluded that either Q was a prime $> p$ or the product of primes each $> p$.

The values of Q for the first eight values of p are given below:

p	Q		
2	3		
3	7		
5	31		
7	211		
11	2311		
13	30031	=	59(509)
17	510511	=	19(97)(277)
19	9699691	=	347(27953)

What happens beyond this? Are any more Q's prime? $p = 17$ and the smallest factor of Q are twin primes. Does this situation of twin primes, or even consecutive primes, occur again? Are there any more cases where the smallest factor of $Q < 2p$? What is the smallest p for which Q has four prime factors? Five prime factors?

With the aid of a computer, some skilled investigator may be able to answer these questions.

Since, for $p > 2$, Q has the form $6k + 1$, the number of its prime factors of the form $6k - 1$ must be even.

NOTES AND COMMENTS

Several of our readers have written in response to *A note on matrix inversion* by A. Polter Geist (September 1973) giving simple legitimate ways to invert Geist's matrix S. There are several such methods. It seems desirable to let interested readers discover them for themselves.

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BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

Materials intended for review should be sent to: Professor Ada Peluso, Department of Mathematics, Hunter College of CUNY, 695 Park Avenue, New York, New York 10021, or to Professor William Wooton, 1495 La Linda Drive, Lake San Marcos, California 92069. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use.*

Mathematics for the Liberal Arts Student. By Fred Richman, Carol Walker, and Robert J. Wisner. Brooks / Cole, Monterey, Cal., 1973. 294 + ix pp. $9.95.

Texts for liberal arts mathematics courses generally fall into either of two categories: books which stress arithmetic and books which stress patterns. For this volume, the authors attempted to use the latter approach without sacrificing the former. By providing an appendix in which the rules of arithmetic are reviewed, the authors permit students who need review to have it without delaying the progress of the more advanced students. This is an excellent idea though the execution did not satisfy the reviewer.

The body of the text, too, seemed somewhat inadequate. Unfortunately, the authors seemed so enamored of their numbers that they lost sight of their audience — students who, at best, tend to be weak in mathematics. These students require several examples of the procedures discussed. In this book, too few examples have been provided.

Just as there must be many examples, there must also be numerous problems carefully designed to lead the students to suitable discoveries. While many problems have been provided in this book, they do not seem to lead the student to make generalizations, that is, to recognize patterns. The problems which have been furnished seem unclear at times and too few answers have been provided at the back of the volume. In some instances, the problems lead to incorrect generalizations or to incomplete ones.

Problems were placed in the book in such a way as to detract from the continuity of the prose. This fact and the apparent change in style from section to section cause the book to be uncomfortable to read. In many instances, the authors seemed to patronize the students.

All-in-all, the reviewer found the book lacking in the attributes necessary to fulfill its purpose.

J. M. DIMSDALE, Orange Coast College, Costa Mesa

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Mathematics: A Creative Art. By Julia Wells Bower. Holden-Day, San Francisco, 1973. xvi + 315 pp. $10.50.

"The course for which this text was written," explains Julia Wells Bower in the preface, "is designed to show mathematics not as a tool or technique but as a creative form of abstract thinking." That the course began in 1954 and dealt with topics from abstract algebra, set theory and geometry reveals two facts about the book.

First, it was long in the making, and in the preface Professor Bower describes its development from a syllabus through a preliminary edition to its present state. The result is a very well written textbook which is a model of meticulous construction. Each subject introduced is explained clearly and concisely, each definition is given when needed and in appropriately exact terms, and exercises are so well integrated with the text that "results found in the exercise at the end of a section are often the basis for the development in the next section" so that high levels of expository vigor and rigor are maintained throughout this volume.

Second, the book treats those same abstract topics with which Professor Bower began her course: abstract algebra, set theory and geometry. The result is a text which says nothing about the usefulness of mathematics in science, engineering, business or government. (In Chapter 7 the half dozen example applications of matrices are very artificial.) And since it does not provide survey coverage of important mathematical topics nor offer historical insights, except for the accurate historical sketch given on geometry in Part III, this work is free to examine in detail the axiomatic implications of a few carefully selected mathematical systems and thereby "to demonstrate the nature and the power of the postulational method." The pedagogic prowess of the book is indicated, I think, by the manner in which it skillfully involves the student in a minute examination of mathematical structures, at first with simple concepts and later with more sophisticated ones. It is by means of such careful student examinations of postulational systems and their alternatives that Professor Bower endeavors to bring him to an appreciation of mathematics as "a creative art."

The book is divided into three parts: I. Algebraic Systems, II. Logic and Sets, and III. Geometry. Part I, comprising slightly more than one-half of the book, is constituted of the following seven chapters: the five-element system: addition; the five-element system: multiplication; the five-element system: further properties; a six-element system, a system of pairs of elements, a second system of pairs of elements, and a system of matrices. Within each system the operations and their properties of closure, commutativity, associativity, etc., are carefully defined and illustrated. By comparing and contrasting similarities and differences between systems and drawing appropriate generalizations, increasingly more sophisticated concepts are introduced. As a result, certain characteristic features of each class of numbers — naturals, integers, rationals, and reals — are noted and briefly examined. Also, the concepts of group, equivalence class and isomorphism are defined and discussed.

The book is so tightly written the author rarely misses an opportunity to capitalize on developments that have come earlier in the text or problem exercises. But after discussing the solution of matrix equations she does, however, fail to exhibit the matrix solution of a set of two simultaneous linear equations in two variables, perhaps because she wished to avoid nonsquare matrices. Instead Professor Bower presents a somewhat cumbersome illustration and statement of the so-called Cramer's Rule for solving such systems by means of determinants. She seems to be unaware of Carl B. Boyer's, *Colin Maclaurin and Cramer's Rule*, Scripta Mathematica, Vol. 27, January, 1966, pp. 377–379. Here Professor Boyer shows that Maclaurin published (posthumously) the so-called Cramer's Rule in 1748, two years prior to Gabriel Cramer's publication of it in 1750. Boyer closes his note with the question: "In view of Maclaurin's priority in publication, and of Cramer's superiority in exposition, might it not be well to call this determinant device neither 'Cramer's Rule' nor 'Maclaurin's Rule,' but rather the 'Maclaurin-Cramer Rule'?"

Part II on Logic and Sets, constituting slightly less than one-fourth of the book, is comprised of two chapters only, one on the algebra of logic and the other on the algebra of sets. Both topics are treated carefully and in greater depth than is usual for a general education mathematics text. In Part III, making up about one-fifth of the book, the author turns away from algebra, which clearly is her first love, to geometry. Here in three chapters, titled introduction to geometry, hyperbolic geometry, and elliptic geometry, the student is given the opportunity to test his mathematical sophistication on new, nonalgebraic material. Once again the presentation is unusually thorough for a work of this kind.

I would recommend this text to those in sympathy with Julia Wells Bower's distinctive approach to the teaching of liberal arts mathematics. The book is designed for use in a variety of one-semester or half-year courses. A separately published Instructor's Manual is available. Finally, it should be noted that the book is well produced and virtually free of printing errors.

RANDALL LONGCORE, York University, Toronto

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

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Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before October 1, 1974.

PROBLEMS

901. *Proposed by Leon Bankoff, Los Angeles, California.*

The sides of any triangle are rational (or integral) only if the ratio of the inradius to the circumradius is rational. Is the converse true?

902. *Proposed by J. Michael McVoy and Anton Glaser, Pennsylvania State University.*

When only the two antigens A and B were known, there were four blood types, corresponding to the four subsets of these antigens. Now that antigen Rh has been brought to light, there are eight blood types. Emergency blood donations are subject to the rule that the donor's set of antigens must be a subset of the recipient's set of antigens. Thus any pair of blood types falls into one of three categories: (i) each owner may donate to the other, the two types being the same; (ii) only one owner may denote to the other; and (iii) neither owner may donate to the other.

Let n be the number of antigens on which blood typing and above rule are based. Category (iii) has 0, 1, and 9 pairs for $n = 1, 2$, and 3 respectively. How many pairs are in (iii) for $n = k$?

903. *Proposed by Alan Wayne, Holiday, Florida.*

Solve the equation

$$TWO \times ZERO = NOTHING$$

in which each letter represents a denary digit, with different letters representing different digits, and such that GOO, ROO, TOO, WOO and ZOO are primes. Is the information about the primes essential for a unique solution?

904. *Proposed by Thomas W. Hill. Jr., Purdue University.*

Show that in an $m \times n$ matrix of real numbers there exists at least one entry that is both maximum in its row and minimum in its column. Do this without using the ordering property of the reals directly.

905. *Proposed by Marlow Sholander, Case Western Reserve University.*

Let Γ be the graph of $y = f(x) = ax^3 + bx^2 + cx + d$. Given only Γ, how does one construct through a point on Γ lines tangent to Γ?

906. *Proposed by Robert Guy, Framingham, Massachusetts.*

Prove that $\prod_{m=1}^{n} m^{2[n/m]} - \mathscr{I}(m) = 1$, where [ ] is the bracket function and $\mathscr{I}(m)$ is the number of divisors of m.

907. *Proposed by Warren Page, New York City Community College.*

The characteristic function of the rationals, although discontinuous at every point of the real line R, is equal almost to a continuous function on R. Is it possible to construct a function discontinuous at every point of R which is not equal almost everywhere to a measurable (Lebesque) function?

Errata. In Q573, Page 231, September 1973, the second radicand should read $k^2 - 16k$.

In Problem 881, page 285, November 1973, all signs should be positive.

Problem 890, proposed by Zalman Usiskin, page 47, January 1974, should read as follows:

"Let a and b be any complex numbers, $a \neq b$. For what a and b does there exist an operation * in C^2 such that $C - \{a\}$ is a group under * with b as the identity?"

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q593. Find the next three numbers in the series: 4, 6, 9, 10, 14, 15, 21, 22, 25, 26, 33, 34, 35, 38.

[*Submitted by Dixon Jones*]

Q594. Show that if n is a prime that does not divide the integer r then $x^n + y^n = z^r$ has a solution in integers.

[*Submitted by Glenn D. James and Charles W. Trigg, jointly*]

Q595. Find the sum of the series $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$

[*Submitted by Michael Golomb*]

Q596. If $x_i > 0$, $\sum_{i=1}^{n} x_i = 1$, $x_{n+1} = x_1$ and $n > 6$, then $\prod_{i=1}^{n} (x_i + x_{i+1})^{-1} = n!$

[*Submitted by Norman Schaumberger*]

Q597. Prove that

$$\frac{(n+1)^{n+1}}{n^n} > \frac{n^n}{(n-1)^{n-1}}$$

for $n = 1, 2, 3, \cdots$ (here $n^n = 1$ for $n = 0$).

[*Submitted by Murray S. Klamkin*]

(Answers on page 178.)

SOLUTIONS

Magic Square

873. [September, 1973] *Proposed by A. G. Bradbury, North Bay, Ontario, Canada.*

In this alphametic, of course, each distinct letter stands for a particular but different digit in the decimal notation. The array of three-digit words corresponds to a rectangular Magic Square. What must the car be?

$$\begin{array}{ccc} & & A \\ N & E & W \\ T & O & Y \\ \hline C & A & R \end{array}$$

Solution by Kenneth M. Wilke, Topeka, Kansas.

Since the only magic squares of order 3 having decimal digits for entries are

7	0	5
2	4	6
3	8	1

and

8	1	6
3	5	7
4	9	2

,

$$\begin{array}{ccc} N & E & W \\ T & O & Y \\ \hline C & A & R \end{array}$$

must be a rotation and/or a reflection of one of these squares. Testing the various possibilities,

$$\begin{array}{ccccccc} C & A & R & = & 8 & 3 & 4 \\ N & E & W & = & 6 & 7 & 2 \\ T & O & Y & = & 1 & 5 & 9 \end{array}$$

is the unique solution.

Also solved by: Leon Bankoff, Los Angeles, California; Gladwin Bartel, Otero Junior College, La Junta, Colorado; Jeffrey H. Baumwell, Whitestone, New York; Gerald E. Bergum, South Dakota State University, Brookings; Timothy L. Bock, Oberlin College, Ohio; Richard L. Breisch, Alamogordo, New Mexico; Robert X. Brennan, Dover, New Jersey; David C. Brook, Seattle Pacific College, Washington; Maxey Brooke, Sweeney, Texas; Evan Coravos and Jeff Gulya (jointly), St. Anselm's College, Manchester, New Hampshire; Charles De Nicola, Richmond Hill, New York; Edward B. Dressman III, Thomas More College, Covington, Kentucky; Donald Elsea, Nashville, Tennessee; George Fabian, Park Forest, Illinois; Marjorie Fitting, San Jose State University, California; Arne Fransén, Research Institute for Defense, Stockholm, Sweden; Harry M. Gehman, Buffalo, New York; John R. Heath, Minneapolis, Minnesota; J. A. H. Hunter, Toronto, Canada; Richard A. Jacobson, Houghton College, New York; Ralph Jones, University of Massachusetts, Amherst; Margaret Kenney, Boston College Mathematics Institute, Chestnut Hill, Massachusetts; Peter A. Lindstrom, Genesee Community College, Batavia, New York; Janice A. McGoldrick, Cranston High School East, Rhode

Island; Joleen Michalowicz, St. Anthony's School, Falls Church, Virginia; Joseph V. Michalowicz, Falls Church, Virginia; Dan Scholten, Wesleyan University, Middletown, Connecticut; Abraham Schwartz, Jamaica, New York; Paul Smith, University of Victoria, Canada; Charles W. Trigg, San Diego, California; Wolf R. Umbach, Rottdorf, Germany; Zalman Usiskin, University of Chicago; R. F. Wardrop, Central Michigan University, Mt. Pleasant; Kenneth M. Wilke, Topeka, Kansas; Dale Woods and Ken Eccher (jointly), Northeast Missouri State University, Kirksville; Kenneth L. Yocom, South Dakota State University; Gene Zirkel, Nassau Community College, Garden City, New York; and the proposer.

Editor's note: Joleen Michalowicz, a fourth grade student, is the youngest solver in my twenty-two years with this MAGAZINE.

Supermagic Square

874. [September, 1973] *Proposed by David Singmaster, London, England.*

An $n \times n$ array is called supermagic if every $n-1 \times n-1$ subarray obtained by removing a column and a row has the same sum. Find all supermagic squares.

Solution by Ralph Jones, University of Massachusetts, Amherst.

The name is terrible, since the only such $n \times n$ array is one with every entry the same. This is trivial for $n = 1$ or 2. So let $n \geqq 3$.

The property in question is preserved when any two columns are interchanged, or any two rows. It can be assumed that the term a_{11} in the first row and first column is the smallest term of the array, and that the first row is a nondecreasing sequence, and that the first column is a nondecreasing sequence.

The property in question is preserved when any constant is subtracted from each term of the array. It can be assumed that $a_{11} = 0$.

Let $R_k(C_m)$ be the sum of the terms of the kth row (mth column). Then $R_k + C_m - a_{km} =$ (constant), and

$$\begin{cases} R_k - a_{km} = R_s - a_{sm} \\ R_k - a_{kt} = R_s - a_{st}. \end{cases}$$

so that $a_{kt} - a_{km} = a_{st} - a_{sm}$ and each two columns differ by a constant. Similarly, each two rows differ by a constant. Let $r_k = a_{k1} - a_{11} = a_{k1}$ and $c_m = a_{1m} - a_{11} = a_{1m}$ be these constants. Thus $a_{km} = r_k + C_m$.

Let $U = \Sigma r_k$, $V = \Sigma c_m$, S be the sum of the $n \times n$ array, and W be the constant sum of each of the $(n-1) \times (n-1)$ subarrays. When W is computed by deleting the kth row and the mth column, $W = (n-1)\ (U+V) - (n-1)\ (r_k + c_m)$. Thus $a_{km} = r_k + c_m =$ (constant), and the claim is established.

Also solved by: Gladwin Bartel, Otero Junior College, La Junta, Colorado; G. E. Bergum, South Dakota State University, Brookings; George Fabian, Park Forest, Illinois; Richard A. Gibbs, Fort Lewis College, Durango, Colorado; M. S. Klamkin, Ford Motor Company, Dearborn, Michigan; Edward T.-H. Wang, Wilfrid Laurier University, Waterloo, Ontario; Kenneth M. Wilke, Topeka, Kansas; Kenneth L. Yocom, South Dakota State University; Gene Zirkel, Nassau Community College, Garden City, New York; and the proposer.

Interpolating Cauchy's Inequality

875. [September, 1973] *Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.*

If $\{a_i\}$, $\{b_i\}$ denote two sequences of positive numbers and n is a positive integer, show that:

$$\sum_i a_i^{2n} \cdot \sum_j b_j^{2n} \geqq \sum_i a_i^{2n-1} b_i \cdot \sum_j a_j b_j^{2n-1} \geqq \cdots \geqq \sum_i a_i^n b_i^n \cdot \sum_j a_j^n b_j^n.$$

Solution by Robert M. Hashway, West Warwick, Rhode Island.

Solution: Since the inequalities are trivially true when either the a_i's or the b_i's are all zero, the a_i's and the b_i's may be either *positive or zero*. Hence, $\{a_i\}$ and $\{b_i\}$ can be made of the same dimension by inserting zeroes. What must be shown is that: if $\{a_i\}$, $\{b_i\}$ are sequences consisting of positive real numbers or zero, and n is a positive number, then

$$\sum_i a_i^{2n-k} b_i^k \sum_j a_j^k b_j^{2n-k} \geqq \sum_i a_i^{2n-k-1} b_i^{k-1} \quad \sum_j a_j^{k-1} b_j^{2n-k-1}$$

where k is an integer such that $n-1 \geqq k \geqq 0$.

Proof: We need to determine if the expression below is positive or zero:

$$\sum_i a_i^{2n-k} b_i^k \sum_j a_j^k b_j^{2n-k} - \sum_i a_i^{2n-k-1} b_i^{n+1} \sum_j b_j^{2n-k-1} a_j^{k+1}.$$

We can easily see that:

$$\sum_i a_i^{2n-k} b_i^k \sum_j a_j^k b_j^{2n-k} = \sum_i a_i^{2n} b_i^{2n} + \sum_i \sum_{j \neq i} a_i^{2n-k} a_j^k b_i^k b_j^{2n-k}$$

and

$$\sum_i a_i^{2n-k-1} b_i^{k-1} \sum_j a_j^{k-1} b_j^{2n-k-1} = \sum_i a_i^{2n} b_i^{2n} + \sum_i \sum_{j \neq i} a_i^{2n-k-1} b_i^{k+1} a_j^{k+1} b_j^{2n-k-1}.$$

Hence, what remains to prove is that,

$$s_k = \sum_i \sum_{j \neq i} (a_i^{2n-k} a_j^k b_i^k b_j^{2n-k} - a_i^{2n-k-1} b_i^{k+1} \quad a_j^{k+1} b_j^{2n-k-1}) \geqq 0.$$

By interchanging the indices i and j, and summing over all j, i, we have:

$$\begin{aligned} s_k = \sum_i \sum_{j<i} (& a_i^{2n-k} a_j^k b_i^k b_j^{2n-k} + a_j^{2n-k} a_i^k b_j^k b_i^{2n-k} \\ & - a_i^{2n-k-1} b_i^{k+1} a_j^{k+1} b_j^{2n-k-1} \\ & - a_j^{2n-k-1} b_j^{k+1} a_i^{k+1} b_i^{2n-k-1}). \end{aligned}$$

By factoring out similar terms, we have:

$$s_k = \sum_i \sum_{j<i} a_j^k b_i^k a_i^k b_j^k (b_i a_j - b_j a_i)((a_j b_i)^m - (a_i b_j)^m)$$

where $m = 2(n - k) - 1$.

Since each term of the series is positive, the result is clear.

Also solved by: J. L. Brown, Jr., Pennsylvania State University; Leon Gerber, St. John's University, Jamaica, New York; M. G. Greening, University of New South Wales, Australia; Robert M. Hashway, West Warwick, Rhode Island; Richard A. Jacobson, Houghton College, Houghton, New York; Alan H. Stein, University of Connecticut; and the proposer.

Arrangements on a Circle

876. [September, 1973] *Proposed by Steven R. Conrad, Benjamin Cardozo High School, Bayside, New York.*

Let $\{a_i\}$ represent a set of n arbitrary real numbers whose sum is positive. Prove that no matter how these numbers are arranged on the circumference of a circle, there always exists at least one of them, to be called a_1, such that if these numbers are subscripted consecutively from 1 to n in a clockwise direction, it is always true that

$$\sum_{i=1}^{k} a_i > 0 \text{ for } k = 1, 2, \cdots n.$$

Solution by N. J. Kuenzi and Bob Prielipp, University of Wisconsin at Oshkosh.

Our solution will employ induction on n. Clearly the desired result holds when $n = 1$ and when $n = 2$. Assume that the result holds for k real numbers, $k \geqq 2$, and consider an arbitrary arrangement on the circumference of a circle of $k+1$ real numbers whose sum is positive. Find a positive number whose sum with the number that follows it is positive. (Such a number must exist because the total sum of the numbers is positive.) If this pair of numbers is replaced by its sum S, an arrangement of k numbers is obtained. Use the induction hypothesis to determine a starting point a_1^*. If $a_1^* \neq S$, let $a_1 = a_1^*$. If $a_1^* = S$, let a_1 be the first member of the pair whose sum was S.

Also solved by Donald Batman, Massachusetts Institute of Technology; Larry I. Bennett and Kenneth L. Yocom (jointly), South Dakota State University; Timothy L. Bock, Oberlin College, Ohio; Richard L. Breisch, Alamogordo, New Mexico; Brother Alfred Brousseau, St. Mary's College, Moraga, California; J. L. Brown, Jr., Pennsylvania State University; Kevin Brown, Mobile, Alabama; Leon Gerber, St. John's University, Jamaica, New York; Michael Gilpin, Michigan Technological University, Houghton; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Robert Gross, B. Cardozo High School, Bayside, New York; F. David Hammer, Stockton State College, New Jersey; Vladimir F. Ivanoff, San Carlos, California; Richard A. Jacobson, Houghton College, New York; Ralph Jones, University of Massachusetts; M. S. Klamkin, Ford Motor Company, Dearborn, Michigan; J. F. Leetch, Bowling Green State University, Ohio; Peter W. Lindstrom, St. Anselm's College, Manchester, New Hampshire; Larry Olson, University of Wisconsin, Wausau; Paul Smith, University of Victoria, Canada; Alan H. Stein, University of Connecticut; Eric Storberg, Southern Illinois University, Edwardsville; Phil Tracy, Liverpool, New York; Edward T.-H. Wang, Wilfrid Laurier University, Ontario, Canada; Kenneth M. Wilke, Topeka, Kansas; Katherine Woerner, Westlake High School, Austin, Texas; and the proposer.

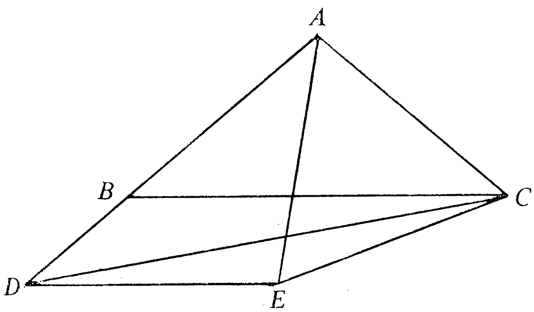
A Cape Town Problem

877. [September, 1973] *Proposed by H. J. Webb, Cape Town University, Republic of South Africa.*

In triangle ABC, $\angle ABC = \angle ACB = 40°$. AB is produced to D such that $AD = BC$. Determine $\angle BCD$.

Solution by J. W. Wilson, Athens, Georgia.

Draw DE parallel to BC such that $DE = AB$. Now since $AD = BC$ and $\angle BDE = \angle ABC = 40°$, then triangle ABC is congruent to triangle EDA. Therefore $AE = AC$, $\angle DAE = 40°$, and so $\angle EAC = 60°$. So triangle AEC is equilateral, and $EC = DE$. Consider triangle CED. The triangle is isosceles with base angles of 10°. But $\angle BCD$ is congruent to $\angle CDE$. Therefore $\angle BCD = 10°$.



Also solved by: Jorge Andres, St. Francis College, Brooklyn, New York; Leon Bankoff, Los Angeles, California; Charles N. Baker, West Liberty State College, West Virginia; Leila Barge, Lounell Snodgrass and Dale Woods (jointly), Northeast Missouri State University, Kirksville; Gladwin Bartel, Otero Junior College, La Junta, Colorado; Jeffrey H. Baumwell, Whitestone, New York; Gerald E. Bergum, South Dakota State University; Martin Berman, Bronx Community College, New York; Melvin Billik, Midland High School, Michigan; Timothy L. Bock, Oberlin College, Ohio; Robert X. Brennan, Dover, New Jersey; Brother Alfred Brousseau, St. Mary's College, Moraga, California; Martin J. Brown, University of Kentucky, Louisville; Scott Brown, West Virginia University, Morgantown; Mannis Charosh, Brooklyn, New York; Romae J. Cormier, Northern Illinois University; Santo Diano, Havertown, Pennsylvania; Ragnar Dybvik, Tingvoll, Norway; George Fabian, Park Forest, Illinois; Arne Fransén, Research Institute of National Defense, Stockholm, Sweden; Ralph Garfield, The College of Insurance, New York City; J. Garfunkel, Flushing, New York; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Robert M. Hashway, West Warwick, Rhode Island; John M. Howell, Littlerock, California; Vladimir F. Ivanoff, San Carlos, California; M. S. Klamkin, Ford Motor Company, Dearborn, Michigan; Lew Kowarski, Morgan State College, Baltimore, Maryland; Sam Kravitz, Mayfield Heights, Ohio; Peter W. Lindstrom, St. Anselm's College, Manchester, New Hampshire; Joseph V. Michalowicz, Falls Church, Virginia; David G. Phillips, Union College, Poughkeepsie, New York; Lawrence A. Ringenberg, Eastern Illinois University; Dan Scholten, Wesleyan University, Middletown, Connecticut; Paul Smith, University of Victoria, Canada; Eric Storberg, Southern Illinois University, Edwardsville; Steven Szabo, Urbana, Illinois; Zalman Usiskin, University of Chicago; Wolf R. Umbach, Rottdorf, Germany; R. F. Wardrop, Central Michigan University, Mt. Pleasant; Kenneth M. Wilke, Topeka, *Kansas; William Wynne-Willson, University of Birmingham, England; Kenneth L. Yocom, South Dakota State University; Gene Zirkel, Nassau Community College, Garden City, New York; and the proposer.*

Miguel Triangles

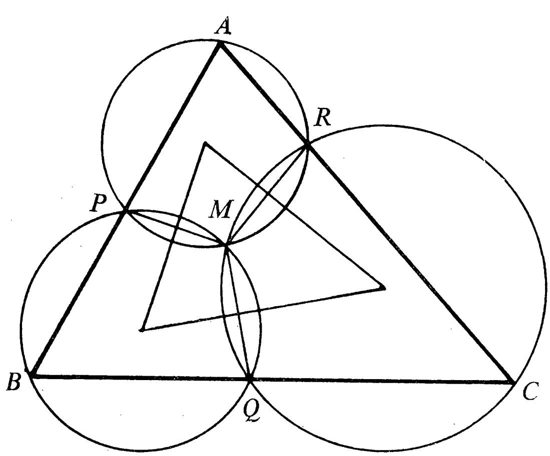
879. [September, 1973] *Proposed by Sally Ringland, Clarion State College, Pennsylvania.*

Given ΔABC, let P be any point on AB, Q be any point on BC and R any point on AC. P, Q and R should not be vertices of the triangle. Consider the circles through A, P and R, through P, B and Q and through Q, C and R. Prove that the centers of the circles determine a triangle similar to ΔABC.

Solution by Leon Bankoff, Los Angeles, California.

It is known that the three constructed circles intersect at the Miguel point M for the triad P, Q, R, with respect to triangle ABC. Furthermore, the lines from the Miguel point to the marked points make equal angles with the respective sides. (See Roger A. Johnson, *Modern Geometry*, Houghton Mifflin, 1929, Page 133, or the same reference in the Dover Reprint, entitled *Advanced Euclidean geometry.*)

Since the lines MP, MQ and MR are common chords of the circles taken in pairs and are perpendicular to the respective connectors of the centers of the circles, it follows that the connectors also make equal angles with the respective sides of triangle ABC taken in the same sense. Consequently the triangle of centers is similar to triangle ABC.



Also solved by: Leon Gerber, St. John's University, Jamaica, New York; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Vladimir F. Ivanoff, San Carlos, California (two solutions); M. S. Klamkin, Ford Motor Company, Dearborn, Michigan; Lew Kowarski, Morgan State College, Baltimore, Maryland; R. F. Wardrop, Central Michigan University, Mt. Pleasant; William Wynne-Willson, University of Birmingham, England; and the proposer.

Comment on Problem 823

823. [January, 1972 and November, 1972] *Proposed by Ira Gessel, Dayton, Ohio.*

Find a function f such that $\Sigma_{k=1}^{m} f(k)[m/k] = m(m+1)/2$.

Comment by Robert S. Stacy, Albuquerque, New Mexico.

To respond in terms of Euler's ϕ-function to the problem as stated, it is appropriate to prove a theorem along the lines of:

(1) If a function f equals Euler's ϕ-function, then for all m, $\Sigma_{k=1}^{m} f(k)[m/k] = m(m+1)/2$.

By assuming the existence of a function with the stated property, the published solution addresses the converse of (1), namely that a function with the property equals ϕ. However, as discussed below, the completeness of this proof is open to question.

A proof of (1) using the central idea of the published solution may be given as follows:

It can be verified quickly that the equation is true for $m = 1$.

Now suppose that for some $m = n$, $\Sigma_{k=1}^{n} f(k)[n/k] = n(n+1)/2$.

Then

$$\sum_{k=1}^{n+1} f(k)[(n+1)/k] - \sum_{k=1}^{n} f(k)[n/k] = \sum_{k=1}^{n+1} f(k)[(n+1)/k] - n(n+1)/2.$$

From this

$$f(n+1) + \sum_{k=1}^{n} f(k)\{[(n+1)/k] - [n/k]\} = \sum_{k=1}^{n+1} f(k)[(n+1)/k] - n(n+1)/2$$

where

$$[(n+1)/k] - [n/k] = \begin{cases} 1 \text{ if } n+1 \text{ is divisible by } k \\ 0 \text{ otherwise.} \end{cases}$$

That is, for all $k < n+1$ and k is a divisor of $n+1$,

$$f(n+1) + \sum_{k} f(k) = \sum_{k=1}^{n+1} f(k)[(n+1)/k] - n(n+1)/2.$$

But $f = \phi$ and therefore the l.h.s. must equal $n+1$. That is,

$$\sum_{k=1}^{n+1} f(k)[(n+1)/k] = n+1+n(n+1)/2 = (n+1)(n+2)/2.$$

Hence the equation is true for $m = n+1$ and therefore for all m.

In the published solution the conclusion is reached that for all k, divisors of m, $\Sigma_k f(k) = m$ and therefore $f = \phi$. Now it is true that $\Sigma_d \phi(d) = m$ is a property of ϕ but can it be proved that the property is unique to ϕ among all the functions on the positive integers? If so, a reference seems appropriate. If not, the final conclusion is not valid.

Comment on Problem 837

837. [May, 1972 and March, 1973] *Proposed by Vladimir F. Ivanoff, San Carlos, California.*

Prove that the altitudes of any triangle bisect the angles of another triangle whose vertices are the feet of the altitudes of the first triangle.

Comment by Mannis Charosh, Brooklyn, New York.

A short proof is possible, omitting the use of trigonometric functions. Refer to the diagram on Page 110 [March, 1973]. Omit the perpendiculars from B' and A', and let P be the intersection of the altitudes. We have $AB'PC'$ and $BA'PC'$ are concyclic. Therefore $\angle PC'B' = PAB'$ and $\angle PC'A = \angle PBA'$. But $\angle PAB' = \angle PBA'$ (each is the complement of $\angle C$). Hence $\angle PC'B' = \angle PC'A'$.

Comment on Problem 850

850. [November, 1972 and September, 1973] *Proposed by Richard Dykstra, University of Missouri-Columbia.*

If $a_0 > 0, 0 < \alpha < 1$, and $a_i = a_{i-1} + a_{i-1}^{\alpha}$, $i = 1,2,3\cdots$ for what values of β will $\sum_{k=1}^{n}(1/a_k)^{\beta}$ converge?

I. *Comment by N. J. Kuenzi, University of Wisconsin at Oshkosh.*

The "solution" for Problem 850 which appears on Pages 236–37 of the September, 1973 issue of this MAGAZINE, is incorrect. The ratio test for series convergence was misquoted and then misused. This error then led to the erroneous conclusion that the series $\sum_{k=1}^{\infty} (1/a_k)^{\beta}$ converges if $\beta > 0$. The solution which I submitted shows that the series diverges whenever $\alpha + \beta \leqq 1$.

The series $\sum_{k=1}^{\infty} (1/a_k)^{\beta}$ converges if $\alpha + \beta > 1$ and diverges if $\alpha + \beta \leqq 1$.

Suppose that $\alpha + \beta \leqq 1$ and that $\beta > 0$. (If $\beta \leqq 0$, the series clearly diverges.) Let m be a positive integer such that $a_0 \leqq m^{1/(1-\alpha)}$. Then $a_1 \leqq m^{1/(1-\alpha)} + m^{\alpha/(1-\alpha)} = m^{\alpha/(1-\alpha)}(m+1) - (m+1)^{\alpha/(1-\alpha)}(m+1) = (m+1)^{1/(1-\alpha)}$. It follows by induction that $a_k \leqq (m+k)^{1/(1-\alpha)}$ and, hence, that $(1/a_k)^{\beta} \geqq (1/(m+k))^{\beta/(1-\alpha)}$. Since $\beta/(1-\alpha) \leqq 1$, the series $\sum_{k=1}^{\infty} (1/k)^{\beta/(1-\alpha)}$ diverges and by the comparison test the series $\sum_{k=1}^{\infty} (1/a_k)^{\beta}$ diverges.

Next suppose that $\alpha + \beta > 1$. Let c be any number such that $1/\beta < c < 1/(1-\alpha)$ and consider the sequence $n - cn^{c(1-\alpha)}$. Since $c(1-\alpha) < 1$, $n - cn^{c(1-\alpha)} \to \infty$ as $n \to \infty$. Hence, there is an integer N such that $n - cn^{c(1-\alpha)} > c$ for all $n \geqq N$. It then follows that $1/n^{c(1-\alpha)} > c/(n-c)$ and $1 + 1/n^{c(1-\alpha)} > 1 + c/(n-c) = \sum_{k=0}^{\infty}(c/n)^k \geqq (1+1/n)^c$. Hence, $n^c + n^{c\alpha} = n^c(1 + 1/n^{c(1-\alpha)}) > (n+1)^c$ for all $n \geqq N$. Since $a_n \to \infty$ as $n \to \infty$, there is an integer m such that $a_m \geqq N^c$. By induction it follows that $a_{m+k} \geqq (N+k)^c$ and, hence, that $(1/a_{m+k})^{\beta} \leqq (1/(N+k))^{\beta c}$. Since $\beta c > 1$, the series $\Sigma_{k=1}^{\infty}(1/k)^{\beta c}$ converges and by the comparison test the series $\Sigma_{k=1}^{\infty}(1/a_k)^{\beta}$ converges.

II. *Editorial comment.*

Set $\alpha = 1 - t$, $\beta = pt$, $b_k = a_k^t$. Then $a_k^{\beta} = b_k^p$, and the recurrence relation for the positive numbers b_k is

$$b_{k+1}/b_k = (a_{k+1}/a_k)^t = (1 + b_k^{-1})^t, \qquad 0 < t < 1.$$

Now for $0 < r < 1$, and t increasing from 0 to 1, the function $((1+r)^t - 1)/t$ increases from $\ln(1+r)$ to r. Hence for $0 < r, t < 1$ we have

$$\ln 2 < \ln(1+r)^{1/r} < ((1+r)^t - 1)/rt < 1.$$

Setting $r = b_k^{-1}$ and multiplying by t yields the inequality

$$t \ln 2 < (b_{k+1}/b_k - 1)b_k = b_{k+1} - b_k < t, \text{ for } b_k > 1.$$

Summing these inequalities from $k = 0$ to $n - 1$ yields

$$nt \ln 2 < b_n - b_0 < nt, \text{ if } b_0 \geqq 1.$$

Since $a_k^{-\beta} = b_k^{-p}$, the series $\Sigma a_k^{-\beta}$ converges if and only if Σk^{-p} converges; namely, when $p > 1$, or $\beta > t = 1 - \alpha$.

Comment on Problem 864

864. [March, 1973 and January, 1974] *Proposed by Charles W. Trigg, San Diego, California.*

In the square array

1	2	3
4	5	8
7	6	9

all but two of the twelve adjacent digit pairs, taken horizontally and vertically, have prime sums.

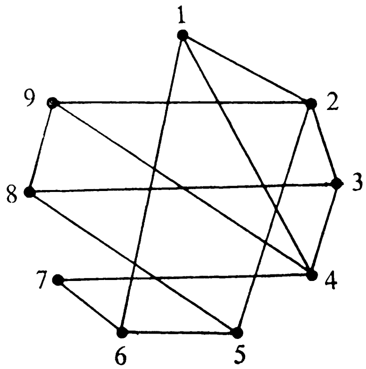
(a) Show that it is impossible to rearrange the digits so that every pair of adjacent digits has a prime sum.

(b) Show that the digits can be rearranged so that each of the twelve sums of adjacent digits is composite, with nine of the composite sums being distinct.

I. *Comment by Eliot W. Collins, New Paltz, New York.*

Construct a graph whose vertices represent the digits from 1 through 9 and whose edges are drawn between every two vertices with a prime sum (see p. 177).

Since the vertex 7 is of order two, 7 must occupy a corner position in the square array with 4 and 6 in adjacent midside positions. Now vertices 2 and 4 are the only ones of order four in the graph, so 2 must be the central digit in the array. But $2 + 4$ is not prime, so an array with all prime sums is impossible.



II. *Comment by the proposer.*

(a) Although there is no completely prime-sum array, there are 10 "almost" complete prime-sum arrays in which only one of the twelve sums is composite. They are

5	2	3	5	2	9	1	2	3	1	2	5	1	2	3
6	1	8	6	1	8	6	5	8	6	3	8	4	9	8
7	4	9	7	4	3	7	4	9	7	4	9	7	6	5

5	8	3	5	8	9	1	2	9	1	2	9	1	2	5
6	1	2	6	1	2	6	5	8	4	3	8	6	9	8
7	4	9	7	4	3	7	4	3	7	6	5	7	4	3

In every case the central element contributes to the composite sum, which is 9 in the first four columns and 15 in the last column. In the three columns on the left, the top array becomes the bottom array upon interchange of two digits. In the two columns on the right, interchange of the digits in two digit-pairs converts the top array into the bottom one.

Furthermore, reading from the upper left, the second, fourth, fifth and six arrays are antimagic, the 8 sums of the rows, columns and unbroken diagonals being distinct.

Comment on Q572

Q572. [May, 1973] Show that if n and k are positive integers then $x^n + y^n = z^{n+1/k}$ always has solutions in integers x, y, z.

[*Submitted by Norman Schaumberger*]

Comment by Murray S. Klamkin, Ford Motor Company.

Since z must be a kth power, we can replace the equation by $x^n + y^n = z^{nk+1}$. One can show more generally that $x^a + y^b = z^c$ always has solutions in integers

x, y, z if a, b, c are positive integers with ab, c relatively prime. Just let $x = 2^{bt} \cdot u^{bc}$, $y = 2^{at} \cdot u^{ac}$, $z = 2^{s}u^{ab}$. Then, $2^{abt+1} = 2^{cs}$.

Since $(ab, c) = 1$, there are infinitely many positive integers, s, t satisfying $abt + 1 = cs$.

ANSWERS

A593. It will be noted that these are the first fourteen numbers that have exactly two prime factors. The next three numbers, then, are 39, 46, and 49.

A594. By Fermat's Theorem, $(r^{n+1} - 1)/n$ is an integer. Then it follows immediately that

$$[2^{(r^{n+1}-1)/n}]^n + [2^{(r^{n+1}-1)/n}]^n = [2^{r^{n-2}}]^r.$$

A595. The sum is

$$\begin{aligned} &1 - 1/3 + 1/5 - 1/7 \cdots \\ &+ 1/2 - 1/4 + 1/6 \cdots \\ &= \pi/4 + 1/2 \ln 2. \end{aligned}$$

A596. Using the arithmetic-geometric mean inequality we get

$$\begin{aligned} \prod_{i=1}^{n} (x_i + x_{i+1}) &\leqq \left[\sum_{i=1}^{n} (x_i + x_{i+1})/n\right]^n \\ &= (2/n)^n, \end{aligned}$$

or

$$\prod_{i=1}^{n} (x_i + x_{i+1})^{-1} \geqq (n/2)^n.$$

The result follows from the familiar inequality $(n/2) > n!$ when $n > 6$.

A597. The inequality can be rewritten as

$$\frac{n+1}{n-1}\left\{1 - \frac{1}{n^2}\right\}^n > 1.$$

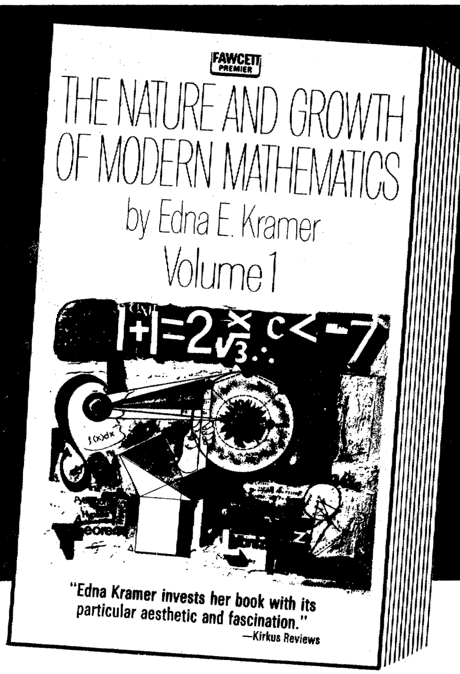
By the Bernoulli's inequality

$$\left\{1 - \frac{1}{n^2}\right\}^n \geqq 1 - \frac{1}{n},$$

whence

$$\frac{n+1}{n-1}\left\{1 - \frac{1}{n^2}\right\}^n \geqq \frac{n+1}{n} > 1.$$

(Quickies on page 167.)



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